Algebraic Characters of Harish-Chandra Modules*

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Abstract

We give a cohomological treatment of a character theory for $(\mathfrak{g}, K)$-modules. This leads to a beautiful formalism extending to large (not necessarily admissible) categories of $(\mathfrak{g}, K)$-modules. Due to results of Hecht, Schmid and Vogan the classical results of Harish-Chandra's global character theory extend to this setting. Apart from discussing the fundamental properties of algebraic characters, our goal is to trace out the merits and limitations of this approach.

Introduction

In a series of fundamental papers [2, 3, 4] Harish-Chandra initiated the theory of $(\mathfrak{g}, K)$-modules and proved the existence and fundamental properties of distribution characters for admissible $(\mathfrak{g}, K)$-modules under the assumption of $Z(\mathfrak{g})$-finiteness and a boundedness condition for the multiplicities of K-types. Harish-Chandra's global characters are a central tool in the study of Harish-Chandra modules.

In this paper we discuss a cohomological algebraic definition of the notion of character essentially for arbitrary $(\mathfrak{g}, K)$-modules. In fact even the admissibility condition can be dropped such that algebraic characters extend to larger categories than analytic global characters do. Of course this introduces complications, and apart from showing the fundamental properties of algebraic characters, our goal is to trace out the merits and limitations of this approach.

That such an algebraic theory exists might not be surprising to the experts, as it is essentially an algebraic formalization of Harish-Chandra's fundamental work.

The connection between characters and cohomology seems to have been observed first by Bott [1] in the finite-dimensional case, and was later refined by Kostant [15], who interpreted the Weyl character formula in terms of Euler characteristics. In a broader context the connection between Harish-Chandra's global characters and cohomology had been conjectured first by Osborne in his thesis [19], later resolved by Hecht and Schmid [6] in the real and Vogan [21] in the θ-stable cases respectively.

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We exploit that, thanks to the work of many people, we can rely on a fairly complete picture concerning the fundamental properties of $\mathfrak{u}$-cohomology and its interplay with global characters. This fills this seemingly bloodless abstract theory with life (cf. Theorem 3.4 below).

Our construction proceeds as follows. For a germane parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$ with nilpotent radical $\mathfrak{u}$ we introduce the notion of $\mathfrak{u}$-admissible pair of categories $(\mathcal{G}, \mathcal{L})$. Here $\mathcal{G}$ is a category of $(\mathfrak{g}, K)$-modules and $\mathcal{L}$ is a category of $(\mathfrak{l}, L \cap K)$-modules for the θ-stable Levi factor $\mathfrak{l}$ of $\mathfrak{q}$. Then $\mathfrak{u}$-admissibility guarantees, apart from another technical assumption, that the $\mathfrak{u}$-cohomology of objects in $\mathcal{G}$ lies in $\mathcal{L}$ and enables us to define the $\mathcal{L}$-valued characters of objects in $\mathcal{G}$ essentially by the Euler characteristic of $\mathfrak{u}$-cohomology, divided by a canonical Weyl denominator.

A practically useful property of $\mathfrak{u}$-cohomology is that it is infinitely additive and sends admissible modules to admissible modules of the Levi factor and preserves $Z(\mathfrak{g})$- resp. $Z(\mathfrak{l})$-finiteness. In particular the categories of modules of finite length resp. discretely decomposable modules are $\mathfrak{u}$-admissible. The categories of admissible modules are also $\mathfrak{u}$-admissible.

Our characters live in a localized Grothendieck group of $\mathcal{L}$. For representations in $\mathcal{G}$ that are already in $\mathcal{L}$ it turns out, a posteriori, that the characters essentially lie in the unlocalized Grothendieck group, in the sense that the representations themselves give a canonical preimage in the unlocalized Grothendieck group, which maps to the character in the localization. In this sense cohomological characters generalize the naive algebraic notion of character.

The cohomological notion of character leads to a beautiful formalism, which follows from purely cohomological arguments (cf. Theorem 1.2 below). In particular it is additive, multiplicative, respects duals, extends both the naive notion of algebraic character as well as Harish-Chandra's, and behaves well in coherent families, i.e. under translation functors. Furthermore it makes (not necessarily admissible) branching problems for restrictions transparent (cf. Proposition 1.5 below). For example it becomes clear in the algebraic picture that Blattner formulas are immediate consequences of character formulas (and in the case of the discrete series even equivalent), provided all irreducible constituents are sufficiently regular, cf. Theorem 4.1 below. A novelty here is that similar statements hold for more general branching problems.

For compact groups it is classical that we recover the classical naive algebraic notion of character by considering a minimal parabolic subalgebra. Along these lines, as is well known, the Weyl character formula in the classical picture is essentially equivalent to (a special case of) Kostant's Theorem on the structure of $\mathfrak{u}$-cohomology. The transitivity of algebraic characters is reflected in the general statement of Kostant's Theorem for not necessarily minimal parabolic subalgebras. As a consequence, from the formal point of view, the case of a maximal parabolic is already enough to prove the Weyl character formula.

By the same transitivity, which is a general principle, fundamental results of Hecht, Schmid and Vogan [6, 21] imply that cohomological characters formally coincide with Harish-Chandra's for modules of finite length and hence characterize composition factors uniquely.

In the world of finite length modules localization does no harm, i.e. the canonical localization map is always injective on the Grothendieck group in question. This situation becomes more involved once we allow discretely decomposable modules, which is a natural setting when approaching branching problems from an algebraic point of view.

Here is an example. Consider the category $\mathcal{C}_{\rm fl}$ of $(\mathfrak{sl}_2, \mathrm{SO}(2))$-modules of finite length, and choose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ a minimal θ-stable parabolic with a Levi decomposition as indicated. Then our characters are multiplicative in the sense that the product of a finite length module M with a finite-dimensional module F has character

$$c_{\mathfrak{q}}(M \otimes_{\mathbf{C}} F) = \mathfrak{q}(M) \cdot \mathfrak{q}(F).$$

This is an idendity in the Grothendieck group of finite length modules of $\mathfrak{l}$, localized at

$$W_{\mathfrak{q}} := 1 - [2\alpha],$$

where -2α is the weight of $\mathfrak{l}$ occuring in $\mathfrak{u}$, and $[-2\alpha]$ being its class. In particular this may be interpreted an identity in a rational function field $\mathbf{Q}(T)$, and as the map $\mathbf{Q}[T] \to \mathbf{Q}(T)$ is injective, we lose no information there, and this is true more generally for any reductive pair.

Now if N is another finite length module, we may be interested in the character of $M \otimes_{\mathbf{C}} N$. However, this is no object in $\mathcal{C}_{\rm fl}$, and in general it is even not in the category of discretely decomposables $\mathcal{C}_{\rm fd}$ with finite multiplicities. However if M and N are both discrete series representations with the property that their SO(2)-types lie in the same $\mathfrak{sl}_2$-Weyl chamber, say are of weight $(2+n) \cdot \alpha$, and $n \geq 0$, then $M \otimes_{\mathbf{C}} N$ lies in $\mathcal{C}_{\rm fd}$. Our formalism therefore says that the identity

$$c_{\mathfrak{q}}(M \otimes_{\mathbf{C}} N) = c_{\mathfrak{q}}(M) \cdot c_{\mathfrak{q}}(N)$$

is true, this time in $\mathbf{Z}[[\alpha, -\alpha]][W_{\mathfrak{q}}^{-1}]$, the module of unbounded Laurent series, localized at the Weyl denominator. Here the natural map

$$\mathbf{Z}[[\alpha, -\alpha]] \quad \to \quad \mathbf{Z}[[\alpha, -\alpha]][W_{\mathfrak{q}}^{-1}]$$

is no more injective, even on the group of Weyl numerators. For example the $\mathfrak{q}$-character of $[D_1] - [D_{-1}]$ lies in the kernel, where $D_{\pm 1}$ denotes the limits of discrete series representation with lowest SO(2)-type of weight $\pm\alpha$. Therefore at this stage we are unable to determine the $\mathfrak{q}$-character (i.e. the composition factors with non-trivial $\mathfrak{q}$-characters) of $M \otimes_{\mathbf{C}} N$ uniquely.

Nonetheless there is a way out. Consider the full subcategory $\mathcal{C}_{\rm fd}^+$ of $\mathcal{C}_{\rm fd}$ of modules subject to the same SO(2)-type condition as M and N. Then it is even true that $M \otimes_{\mathbf{C}} N$ is an object in $\mathcal{C}_{\rm fd}^+$, which is easily seen by solving the branching problem for SO(2), and even though the above localization map is not injective, it is easy to see in this example that the map $c_{\mathfrak{q}}$ is injective when considered as a map from the Grothendieck group of $\mathcal{C}_{\rm fd}^+$ to the above localization. Therefore we may indeed solve the above branching problem:

$$c_{\mathfrak{q}}(D_m \otimes_{\mathbf{C}} D_n) = c_{\mathfrak{q}}(D_m) \cdot c_{\mathfrak{q}}(D_n) = \frac{[m\alpha]}{1 - [2\alpha]} \cdot \frac{[n\alpha]}{1 - [2\alpha]} =$$

$$\sum_{k=0}^{\infty} \frac{[(m+n+2k)\cdot\alpha]}{1-[2\alpha]} = \sum_{k=0}^{\infty} c_{\mathfrak{q}}(D_{m+n+2k}),$$

with the notation D_m for the discrete series of lowest SO(2)-type $m \geq 2$ as above.

In summary we reduced this branching problem to the following two statements:

(C) *Containedness:* The restricted module $M \otimes_{\mathbf{C}} N$ lies in $\mathcal{C}_{\mathrm{fd}}^{+}$.

(I) *Injectivity:* The kernel of $c_{\mathfrak{q}}$ is trivial on the Grothendieck group of $\mathcal{C}_{\mathrm{fd}}^{+}$.

In the second part of this paper we present a general approach to the construction of a large category $\mathcal{C}^{+}$ satisfying (I), starting from a category $\mathcal{C} \subseteq \mathcal{C}_{\mathrm{fd}}$, and at the same times gives a criterion for checking (C) for objects in $\mathcal{C}$.

However in general the formulation of (C) and (I) is more involved, as one single $\mathfrak{q}$ is no more sufficient, and in this sense (C) and (I) should be understood as statements about a collection of characters for (all) different classes of parabolics.

As a complementary case, consider for example the case where $\pi_{\lambda}^{\pm}$ is a principal series representations of $(\mathfrak{sl}_2, \mathrm{SO}(2))$, and we are interested in its restriction to SO(2). Then we still have the identity

$$c_{\mathfrak{q}}(\pi_{\lambda}^{\pm}) = \iota(\pi_{\lambda}^{\pm}),$$

in the appropriate localization, where ι denotes the restriction to SO(2). Then we know that the left hand side vanishes, and so does the right hand side — in the localization. Therefore we know that the right hand side lies *in* the kernel of the localization map. Yet this kernel may be explicitly computed (which is a simple exercise here), and the following two assertions allow us to determine the decomposition:

(B) *Boundedness:* Multiplicities of the SO(2)-types occuring in $\pi_{\lambda}^{\pm}$ is bounded by a constant.

(S) *Sample:* The multiplicities of $0\cdot\alpha$ and $1\cdot\alpha$ in $\pi_{\lambda}^{\pm}$ are known.

Then we may conclude that

$$\iota(\pi_{\lambda}^{\pm}) = \sum_{k\in\mathbf{Z}} [(2k+\delta)\cdot\alpha]$$

where $\delta\cdot\alpha$ is the weight occuring in $\pi_{\lambda}^{\pm}$.

Note that $0\cdot\alpha$ and $1\cdot\alpha$ both do *not* occur in a discretely decomposable $(\mathfrak{sl}_2, \mathrm{SO}(2))$-module M if and only if all its composition factors belong to the discrete series. This is the criterion we have in mind to check (C) above. It is easy to see that for this enlarged category (I) still holds, and it is a maximal subcategory of π_{fd} satisfying this property.

The philosophy behind this example is that the $\mathfrak{q}$-character in the localization plus the information the sample provides is precisely what we need to solve our branching problem (here a Blattner formula) for a general input.

Thus we are naturally led to study the kernels of the localization maps, and so far we are only able to treat the absolute case, which nonetheless is the most important one from the classical perspective. We show that vanishing in the localization forces certain simple symmetries in the character, and those yield the existence of certain irreducible constituents which forms the sample set for (S).

In general condition (B) should read *bounded by a polynomial of fixed degree in the norm of the infinitesimal character*, and the sample in (S) depends on this degree, and is usually not finite (cf. Theorem 5.1 and Corollary 6.7).

A prototypical example for (B) is Harish-Chandra's bound for the multiplicities of K-types in finite length representations — they are bounded by the square of their dimensions, the latter in turn being explicitly computable via the Weyl dimension formula departing from the infinitesimal character.

As Harish-Chandra has shown, this bound is sufficient for the existence of his global character, but also necessary.

The localization problem is analogous to the well known classical situation, where the restriction of the global character may vanish on the regular elements, thus making it difficult to extract the desired information.

In the context of the restriction of a finite length module to a maximal compact subgroup, the vanishing of the restriction of the $\mathfrak{q}$-character to $L \cap K$ for a (not necessarily minimal) θ-stable parabolic $\mathfrak{q}$ (which may be thought of as the generic vanishing of the $\mathfrak{u}$-cohomology), implies the existence of a K-type which is not sufficiently regular, in the sense of condition (6) below. On the other hand if we know that all those 'not sufficiently regular' K-types do *not* occur, or more generally, if we know their multiplicities, then we can read off the corresponding Blattner formula from the character of the non-restricted representation, cf. Theorem 4.1 below.

From the algebraic perspective a relative treatment would be desirable, as this would circumvent the vanishing problem, in the same way as David Vogan's approach to minimal K-types [20] avoids this by considering non-minimal parabolic subalgebras whenever necessary. Our approach is similar yet different, as we rely on the same spectral sequence, but do not focus on a particular K-type.

In the context of a general branching problem, the above observations lead us to the following slightly more effective version of Kobayashi's Conjecture C in [13].

Conjecture 0.1. *Let (G, G') be a semisimple symmetric pair, and $\pi \in \hat{G}$ an irreducible unitary representation of G. Assume that the restriction of π to G' is infinitesimally discretely decomposable, then the dimension*

$$\dim \operatorname{Hom}_{G'}(\tau, \pi|_{G'}), \quad \tau \in \hat{G}'$$

is finite and grows at most polynomially in the norm of the infinitesimal character of τ.

As the character formula for the Zuckerman-Vogan cohomological induction modules $A_{\mathfrak{q}}(\lambda)$ is known, we are optimistic that our approach may be applied to produce more evidence towards Kobayashi's Multiplicity-Free Conjecture and also its relation to the virtually symmetric type, cf. Conjectures 4.2 and 4.3 in [14], at least in the infinitesimally discretely decomposable cases. We hope to come back to this in the future.

Our paper is organized as follows. In a rather long zeroth section we collect well known facts about certain categories of $(\mathfrak{g}, K)$-modules and their cohomology and extend them to discretely decomposable modules whenever possible. In the first section we introduce the abstract notion of cohomological characters. In the second section we translate known results about the $\mathfrak{u}$-cohomology into this setting and show that translation functors essentially commute with characters, also allowing appropriate categories of discretely decomposable modules. In the third section we give applications of the theory and show that algebraic characters determine composition factors of finite length modules uniquely. In section four we treat the problem of reading off Blattner formulas from character formulas, and in section five we generalize our results to discretely decomposable modules with polynomial multiplicity bounds. In the sixth section we examine the kernels of the localization maps, which is relevant for the study of Blattner formulas and more generally discretely decomposable modules from the previous two sections.

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Notation and terminology

The beautiful monograph [9] contains most of the basic notions and results we need.

Reductive pairs

Throughout the paper we fix a reductive pair $(\mathfrak{g}, K)$ where $\mathfrak{g}$ is the complexified Lie algebra of a real Lie algebra $\mathfrak{g}_0$ and K is a maximally compact subgroup in a reductive Lie group G with Lie algebra $\mathfrak{g}_0$. The group G then has finitely many components and we denote G^0 the connected component of the identity in G. We write $\mathfrak{k} \subseteq \mathfrak{g}$ for the complexification of the Lie algebra $\mathfrak{k}_0 \subseteq \mathfrak{g}_0$ of K, $U(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$, and $Z(\mathfrak{g})$ for its center.

There is a natural dictionary between the theory of such reductive groups G and reductive pairs $(\mathfrak{g}, K)$, i.e. for a reductive pair there is a unique G and

vice versa, preserving finite-dimensional representations, cf. [9, Chap. IV]. The group G comes with a Cartan involution θ, which is stricly speaking also part of the datum $(\mathfrak{g}, K)$, as is the invariant bilinear form coming from G and the real Lie algebra $\mathfrak{g}_0$.

We assume that our parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ are always germane, i.e. they possess a θ-stable Levi factor $\mathfrak{l}$ which is the complexification of the Lie algebra $\mathfrak{l}_0$ of the closed reductive subgroup $L \subseteq G$ given by the intersection of the normalizers of $\mathfrak{q}$ and $\theta(\mathfrak{q})$ in G. In this context Levi factors and Cartan subalgebras are always assumed θ-stable. The same terminology applies to parabolic and Cartan pairs. In particular for a Cartan subpair $(\mathfrak{h}, T)$ we have a corresponding subgroup $H \subseteq G$ with $T = H \cap K$. The set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ is denote by $\Delta(\mathfrak{g}, \mathfrak{h})$, and $\rho(\mathfrak{u})$ denotes the half sum of the weights of $\mathfrak{h}$ in $\mathfrak{u}$ ($\mathfrak{h}$ is always clear from the context).

If $\mathfrak{l}$ is an abelian Lie algebra and $\lambda \in \mathfrak{l}^*$ is a character, we write $\mathbf{C}_\lambda$ for the one-dimensional representation space of λ. The trivial representation is denoted simply by $\mathbf{C}$. The reader familiar with the classical picture hopefully accepts our apologies for our consequent ignorance of the classical analytic notation.

We tend to ignore the action of the component group in most of our considerations because it can be recovered from $\mathfrak{u}$-cohomology. In this context the reader might consult Chap. IV Sec. 2 in [9] for an account of Cartan-Weyl's highest weight theory for disconnected groups and Sec. 8 of loc. cit. for infinitesimal characters. However in applications the component groups may pose nontrivial problems that need to be dealt with.

$(\mathfrak{g}, K)$-modules

If X is an irreducible $(\mathfrak{g}, K)$-module, then it is admissible [17]. A theorem of Dixmier says that X then has an infinitesimal character. Hence if X is of finite length, it is necessarily admissible and $Z(\mathfrak{g})$-finite. If X is a $\mathfrak{g}$-module with a compatible action of K, we write X_K for the subspace of K-finite vectors. Then the functor $(\cdot)_K$ is left exact, but not exact in general.

All $(\mathfrak{g}, K)$-modules are assumed to be locally K-finite. We denote by $\mathcal{C}(\mathfrak{g}, K)$ resp. $\mathcal{C}_{\mathrm{a}}(\mathfrak{g}, K)$ resp. $\mathcal{C}_{\mathrm{fl}}(\mathfrak{g}, K)$ the categories of all resp. admissible resp. finitely generated admissible $(\mathfrak{g}, K)$-modules. Note that the latter category conincides with the categories of modules of finite length resp. the category of admissible $Z(\mathfrak{g})$-finite modules.

We write $\mathcal{C}_{\mathrm{d}}(\mathfrak{g}, K)$ for the category of discretely decomposable modules as introduced by Kobayashi [11, Definition 1.1], i.e. modules that are direct limits of finite length modules. This category is a full abelian subcategory of $\mathcal{C}(\mathfrak{g}, K)$. We introduce another category $\mathcal{C}_{\mathrm{f}}(\mathfrak{g}, K)$ as the full subcategory of $(\mathfrak{g}, K)$-modules X with the property that the multiplicity $m_Y(X)$ of any irreducible Y in X is finite. Then this category is again abelian and we denote by $\mathcal{C}_{\mathrm{fd}}(\mathfrak{g}, K)$ the intersection of the latter category with the category of discretely decomposable modules.

We write $K_?(\mathfrak{g}, K)$, $? \in \{\mathrm{f}, \mathrm{a}, \mathrm{fl}, \mathrm{fd}\}$ for the corresponding Grothendieck group of $\mathcal{C}_?(\mathfrak{g}, K)$. We write $[X]$ for the class of X in $K_?(\mathfrak{g}, K)$. It is crucial that the

addition law in $K_?(\mathfrak{g}, K)$ is only *finite*, i.e. comes from splitting short exact sequences. In $\mathcal{C}_?(\mathfrak{g}, K)$, $? \in \{-, \mathrm{fl}\}$ we have a duality sending X to its K-finite dual X^*. Being exact this duality naturally extends to $K_?(\mathfrak{g}, K)$ where $? \in \{-, \mathrm{fl}\}$ (the Grothendieck group being trivial for $? = -$).

By the above, for fixed Y the numbers $m_X(Y)$ have a natural continuation to $K_?(\mathfrak{g}, K)$ that is additive in the variable X and satisfies $m_Y([X]) = m_Y(X)$. As any non-zero X in $\mathcal{C}_?(\mathfrak{g}, K)$, $? \in \{\mathrm{a}, \mathrm{fl}, \mathrm{fd}\}$ has a non-zero composition factor with finite multiplicity, we see that in the respective groups $[X]$ is zero if and only if X is zero. Furthermore $[X] = [Y]$ implies that X and Y have the same composition factors. In that sense the Grothendieck group $K_?(\mathfrak{g}, K)$ is well behaved. We remark that as restriction along a map of reductive pairs $(\mathfrak{l}, L \cap K) \to (\mathfrak{g}, K)$ is exact it descends to the Grothendieck groups.

As $\mathcal{C}_?(\mathfrak{g}, K)$ is not closed under tensor products, we only have a partially defined commutative multiplication which is (finitely) distributive in the obvious way. Localization is well behaved in the following sense. Writing $\mathbf{C}$ for the trivial representation, we get $[\mathbf{C}] = 1$, hence a multiplicative unit exists in $K_?(\mathfrak{g}, K)$. If an element $0 \neq D \in K_?(\mathfrak{g}, K)$ can be multiplied with any element $C \in K_?(\mathfrak{g}, K)$ then the localization $K_?(\mathfrak{g}, K)[D^{-1}]$ is well defined. Generally we can localize at any non-zero linear combination of one-dimensional representations. In the case $K_{\mathrm{fl}}(\mathfrak{g}, K)$ we can localize at any non-zero linear combination of finite-dimensional representations. Due to a result of Kostant [16], [9, Theorem 7.133], this is also true in $\mathcal{C}_{\mathrm{a,fd}}(\mathfrak{g}, K)$. As we may have zero divisors the canonical map $K_?(\mathfrak{g}, K) \to K_?(\mathfrak{g}, K)[D^{-1}]$ is usually far from being injective.

We need the following

Lemma 0.2. *Let $P : \mathcal{C}(\mathfrak{g}, K) \to \mathcal{C}(\mathfrak{g}', K')$ be left adjoint to an exact covariant functor $F : \mathcal{C}(\mathfrak{g}', K') \to \mathcal{C}(\mathfrak{g}, K)$. Then the left derived functors of P commute with direct limits.*

Proof. As P is a left adjoint, it commutes with direct limits. Hence we have for the q-th left derived functors

$$L^q(\varinjlim P)(X_i) = L^q(P \varinjlim)(X_i). \tag{1}$$

It is enough to show the existence of two Grothendieck spectral sequences, one converging to the left hand side, one to the right hand side. Those spectral sequences will collapse by the exactness of direct limits. For the left hand side, nothing is to show as any object is acyclic for the direct limit. For the right hand side we can choose for each object X_i a resolution of standard projectives in the sense of [9, Section II.2]. As the construction of these projectives proceeds by production, which commutes with direct limits, we see that the direct limit of a standard projective is (again a standard) projective. Hence we have a Grothendieck spectral sequence

$$(L^{-p}I)L^{-q} \varprojlim X_i \implies L^{-p-q}(I \varprojlim)(X_i).$$

The edge morphisms of the two spectral sequences yield isomorphisms

$$(L^{-q}P) \varprojlim X_i \cong L^{-q}(P \varprojlim)(X_i) \cong L^{-q}(\varprojlim P)(X_i) \cong \varprojlim L^{-q}P(X_i).$$

This proves the claim. $\square$

Corollary 0.3. *Taking homology as considered below commutes with direct limits. More generally the* Ext*-functors commute with direct limits in the first argument.*

Proposition 0.4 (Wigner's Lemma)**.** *Let χ be an infinitesimal character of a $(\mathfrak{g}, K)$-module X and let Y be a discretely decomposable $(\mathfrak{g}, K)$-module whose composition factors have infinitesimal characters $\neq \chi$. Then*

$$\mathrm{Ext}^n_{\mathfrak{g},K}(Y, X) = 0 \tag{2}$$

for all n.

Proof. This is a consequence of the classical Wigner Lemma (Proposition 7.212 in loc. cit.) which says in our setting that if we write $Y = \varinjlim Y_i$ with Y_i of finite length then $\mathrm{Ext}^n_{\mathfrak{g},K}(Y_i, X) = 0$. As the functor $\mathrm{Ext}^n_{\mathfrak{g},K}(X, \cdot)$ commutes with injective limits by Lemma 0.2, the identity (2) follows. $\square$

As a consequence of Yoneda's description of $\mathrm{Ext}^n_{\mathfrak{g},K}(Y, X)$ as the group of classes of n-extensions of X and Y Wigner's Lemma tells us that any discretely decomposable module Y decomposes into the direct sum of its χ-primary components, where χ runs through the infinitesimal characters. In particular χ-primary components are well defined for discretely decomposable modules.

We have an explicit description of the projection p_χ onto the χ-primary component. Any element $y \in Y$ has a preimage y_i in some Y_i. We can consider the projection p_i of Y_i to its χ-primary component (cf. Proposition 7.20 in loc. cit.). Then the elements $p_i(y_i)$ can be assumed to be compatible elements of the directed system of the $p_i(Y_i)$ and hence their limit is well defined as an element of the injective limit of the χ-primary components of the Y_i.

Corollary 0.5. *For any discretely decomposable $(\mathfrak{g}, K)$-module X there is a canonical decomposition*

$$X \cong \bigoplus_{\chi} X_\chi \tag{3}$$

where χ ranges over the infinitesimal characters of composition factors of X and X_χ is the χ-primary component. Furthermore X_χ is the direct limit of the χ-primary components $(X_i)_\chi$ of the X_i and X_χ is of finite length if X lies in $\mathcal{C}_{\mathrm{fd}}(\mathfrak{g}, K)$.

Proof. The decomposition (3) follows for each finite level from Wigner's Lemma. As direct summs and direct limits commute, (3) holds as stated. The statement about finite length follows from Harish-Chandra's celebrated theorem that, up to isomorphy, there are only finitely many irreducible $(\mathfrak{g}, K)$-modules sharing the same infinitesimal character. $\square$

Lie algebra cohomology

For the convenience of the reader we recall known facts about Lie algebra cohomology. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{u} \subseteq \mathfrak{g}$ a subalgebra and V a $\mathfrak{g}$-module, naturally considered as a $\mathfrak{u}$-module as well.

The cohomology

$$H^\bullet(\mathfrak{u}, V)$$

may be calculated from the finite-dimensional standard complex

$$\mathrm{Hom}_{\mathbf{C}}(\bigwedge^\bullet \mathfrak{u}, V)$$

whose differential is known explicitly. Dually the homology

$$H_\bullet(\mathfrak{u}, V)$$

may be calculated from the finite-dimensional standard complex

$$(\bigwedge^\bullet \mathfrak{u}) \otimes_{\mathbf{C}} V,$$

again with explicit differential. We remark that by hard duality [9, Corollary 3.8] we have natural isomorphisms

$$H_q(\mathfrak{u}, V \otimes_{\mathbf{C}} (\bigwedge^{\mathrm{top}} \mathfrak{u})^*) \cong H^{\mathrm{top}-q}(\mathfrak{u}, V).$$

In particular homology *and* cohomology both commute with direct limits.

Now assume that $\mathfrak{g}$ is complex reductive, and that $\mathfrak{q} \subseteq \mathfrak{g}$ is a parabolic subalgebra with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, $\mathfrak{l}$ being a Levi factor and $\mathfrak{u}$ the nilpotent radical. Then we have a natural action of $\mathfrak{l}$ on the above standard complex. The differential d of this complex turns out to be $\mathfrak{l}$-linear.

Let $\mathfrak{h} \subseteq \mathfrak{l}$ be a Cartan subalgebra. We identify characters of $Z(\mathfrak{l})$ via the Harish-Chandra map with characters of $U(\mathfrak{h})^{W(\mathfrak{l},\mathfrak{h})}$.

Proposition 0.6 (Theorem 7.56 of loc. cit.)**.** *Let X be a discretely decomposable (resp. $Z(\mathfrak{g})$-finite) $\mathfrak{g}$-module. Then $H^q(\mathfrak{u}, V)$ is a discretely decomposable (resp. $Z(\mathfrak{l})$-finite) $\mathfrak{l}$-module and if its χ-primary component for the action of $Z(\mathfrak{l})$ is non-zero then*

$$\chi = \chi_\nu$$

where $\nu = w\lambda - \frac{1}{2}\Delta(\mathfrak{n}, \mathfrak{h})$ with some $w \in W(\mathfrak{g}, \mathfrak{h})$ and the χ_λ-primary component of V is non-zero.

Proof. As modules of finite length are $Z(\mathfrak{g})$-finite, this easily reduces to the case of a $Z(\mathfrak{g})$-finite module V, which in turn is discussed in Theorem 7.56 in [9]. ☐

Let us now return to the general setting, i.e. $(\mathfrak{g}, K)$-modules V, where $(\mathfrak{g}, K)$ is a reductive pair. Similarly $(\mathfrak{l}, L \cap K)$ is the reductive pair associated to the Levi factor of a germane parabolic subalgebra $\mathfrak{q}$. We have natural actions of

$(\mathfrak{l}, L\cap K)$ on the above standard complex, which descends to cohomology. This action is natural in the following sense. Denote $p : (\mathfrak{q}, L\cap K) \to (\mathfrak{l}, L\cap K)$ the canonical projection. Then p induces an exact forgetful functor $\mathcal{F}(p) : \mathcal{C}(\mathfrak{l}, L\cap K) \to \mathcal{C}(\mathfrak{q}, L\cap K)$. It turns out that this functor has a right adjoint $I(p)$, whose composition with the forgetful functor along $(\mathfrak{q}, L\cap K) \to (\mathfrak{g}, K)$ furnishes a left exact functor $H^0 : \mathcal{C}(\mathfrak{g}, K) \to \mathcal{C}(\mathfrak{l}, L\cap K)$. The right derived functors of H^0 are naturally isomorphic to $H^q(\mathfrak{u}, \cdot)$ as universal δ-functors.

Proposition 0.7 (Corollary 5.140 of loc. cit.)**.** *If X is an admissible $(\mathfrak{g}, K)$-module, then $H^q(\mathfrak{u}, X)$ is an admissible $(\mathfrak{l}, L\cap K)$-module.*

Corollary 0.8. *If X is a finite length $(\mathfrak{g}, K)$-module, then $H^q(\mathfrak{u}, X)$ is of finite length as $(\mathfrak{l}, L\cap K)$-module.*

Proof. An admissible module is of finite length if and only if it is $Z(\mathfrak{g})$-finite. Proposition 0.6 concludes the proof. □

Proposition 0.9 (Künneth Formula)**.** *Let V and W be two $(\mathfrak{g}, K)$-modules, then*

$$\bigoplus_{p+q=n} H^p(\mathfrak{u}, V)\otimes H^q(\mathfrak{u}, W) \cong H^n(\mathfrak{u}\times\mathfrak{u}, V\otimes_{\mathbf{C}} W),$$

in the category of $(\mathfrak{l}\times\mathfrak{l}, L\cap K\times L\cap K)$-modules.

Proof. The proof is standard. □

Theorem 0.10 (Hochschild-Serre [8])**.** *Let $\mathfrak{g}$ be an arbitrary complex Lie algebra, $\mathfrak{h}\subseteq\mathfrak{g}$ be a Lie subalgebra, V a $\mathfrak{g}$-module that we also consider as a $\mathfrak{h}$-module. We have a cohomological spectral sequence with*

$$E_1^{p,q} = \bigwedge^p (\mathfrak{g}/\mathfrak{h})^* \otimes H^q(\mathfrak{h}, V),$$

and

$$E_1^{p,q} \Longrightarrow H^{p+q}(\mathfrak{g}, V).$$

In all situations that we are interested in, this spectral sequence eventually respects the additional module structures on cohomology, similarly to the Künneth formula.

For compact connected G and $H = T$ a maximal torus we have the fundamental

Theorem 0.11 (Special case of Kostant's Theorem [15], [9, Theorem 4.135])**.** *Let G be compact connected, V be an irreducible finite-dimensional representation of G of highest weight λ with respect to T. Then the T-module $H^q(\mathfrak{u}, V)$ decomposes into one-dimensional spaces with weights*

$$w(\lambda+\rho(\mathfrak{u}))-\rho(\mathfrak{u}),$$

with $w\in W(\mathfrak{g}, \mathfrak{t})$ of length $\ell(w) = q$, each occuring with multiplicity one.

This theorem has a natural generalization to arbitrary parapolic $\mathfrak{u}\subseteq\mathfrak{g}$, also given by Kostant.

1 Algebraic characters

In this section $(\mathfrak{g}, K)$ denotes a reductive pair, $(\mathfrak{q}, L\cap K)$ is a germane parabolic subpair with Levi factor $(\mathfrak{l}, L\cap K)$ and unipotent radical $\mathfrak{u}$.

We say that a pair $(\mathcal{G},\mathcal{L})$ of two full abelian subcategories $\mathcal{G}$ and $\mathcal{L}$ of $\mathcal{C}(\mathfrak{g}, K)$ resp. $\mathcal{C}(\mathfrak{l}, L\cap K)$ is $\mathfrak{u}$-*admissible* if the following three conditions are satisfied:

(i) The category $\mathcal{G}$ contains the trivial representation and the category $\mathcal{L}$ contains the objects
$$\bigwedge^q \mathfrak{u}^*$$
for all $q \in \mathbf{Z}$.

(ii) The category $\mathcal{L}$ is closed under tensorization with the modules
$$\bigwedge^q \mathfrak{u}^*$$
for any $q \in \mathbf{Z}$.

(iii) The $\mathfrak{u}$-cohomology of any object in $\mathcal{G}$ lies in $\mathcal{L}$.

If the pair of categories satisfies one of the following two conditions, we say that the pair is *multiplicative* resp. *has duality.*

(iv) If for two objects V and W in $\mathcal{G}$ the tensor product $V\otimes W$ lies in $\mathcal{G}$, then for any $p, q\in\mathbf{Z}$
$$H^p(\mathfrak{u}, V)\otimes H^q(\mathfrak{u}, W)$$
is an object in $\mathcal{L}$.

(v) If for an object V of finite length in $\mathcal{G}$ the dual V^* lies in $\mathcal{G}$, then for any $p\in\mathbf{Z}$ the dual
$$H^p(\mathfrak{u}, V)^*$$
of cohomology is an object in $\mathcal{L}$.

Proposition 1.1. *The pairs* $(\mathcal{C}(\mathfrak{g}, K), \mathcal{C}(\mathfrak{l}, L\cap K))$*, and* $(\mathcal{C}_?(\mathfrak{g}, K), \mathcal{C}_?(\mathfrak{l}, L\cap K))$ *for* $? \in \{\mathrm{a}, \mathrm{d}, \mathrm{fd}, \mathrm{fl}\}$ *are always* $\mathfrak{u}$*-admissible and they are multiplicative and have duality whenever* $(\mathfrak{l}, L\cap K)$ *is a Cartan subpair.*

We conjecture that we get multiplicativity and duality also in the non-minimal cases.

Proof. The $\mathfrak{u}$-admissibility easily follows from the results in the previous section except for $? = \mathrm{fd}$. That this case also yields a $\mathfrak{u}$-admissible pair is a bit subtle and results from Proposition 0.6 and Corollary 0.5, together with the observation that $\mathcal{C}_{\mathrm{fd}}(\mathfrak{g}, K)$ is closed under tensorization with finite-dimensional representations, which follows from a Theorem of Kostant [16], [9, Theorem 7.133]. This theorem tells us that, given a discretely decomposable X in $\mathcal{C}_{\mathrm{df}}(\mathfrak{g}, K)$ and a finite-dimensional representation W, then for a given irreducible Z there are

only finitely many composition factors Z' of X such that a given irreducible Z occurs in $Z' \otimes_{\mathbf{C}} W$. This shows then $\mathfrak{u}$-admissibility for $? = \mathrm{fd}$.

Finally if $(\mathfrak{l}, L \cap K)$ is a Cartan subpair, $(\mathfrak{l}, L \cap K)$-modules of finite length are finite-dimensional and split into one-dimensional representations. Therefore our $\mathfrak{u}$-admissible pairs are multiplicative and have duality in for $? = \mathrm{a}, \mathrm{d}, \mathrm{fd}, \mathrm{fl}$. $\square$

We write $K(\mathcal{G})$ for the Grothendieck group of the category $\mathcal{G}$. If $(\mathcal{G}, \mathcal{L})$ is $\mathfrak{u}$-admissible, then by axiom 3 the long exact sequence of cohomology furnishes a group homomorphism

$$H : K(\mathcal{G}) \to K(\mathcal{L}),$$
$$[V] \mapsto \sum_q (-1)^q [H^q(\mathfrak{u}, V)],$$

as V is always of finite cohomological dimension. We define the Weyl denominator

$$W_{\mathfrak{q}} := H([\mathbf{C}]) = \sum_q (-1)^q [\bigwedge^q \mathfrak{u}^*].$$

Then the $\mathfrak{u}$-admissibility guarantees that any element in $K(\mathcal{L})$ may be multiplied with $W_{\mathfrak{q}}$.

For applications we cannot always and usually don't have to work with the full Grothendieck group in the image. Therefore we introduce the notion of $\mathfrak{u}$-*admissible quadruple*

$$\mathcal{Q} = (\mathcal{G}, \mathcal{L}, K_{\mathcal{G}}, K_{\mathcal{L}})$$

as follows. $K_{\mathcal{G}}$ resp. $K_{\mathcal{L}}$ are subgroups of $K(\mathcal{G})$ resp. $K(\mathcal{L})$, subject to the following conditions:

(vi) $H(K_{\mathcal{G}}) \subseteq K_{\mathcal{L}}$

(vii) $[\mathbf{C}] \in K_{\mathcal{G}}$ and $[\mathbf{C}] \in K_{\mathcal{L}}$

(viii) for every $C \in K_{\mathcal{L}}$ we have

$$C \cdot H([\mathbf{C}]) \in K_{\mathcal{L}}.$$

Furthermore, we say that the quadruple is *multiplicative* if the pair $(\mathcal{G}, \mathcal{L})$ is multiplicative and if furthermore for any two elements $X, Y \in K_{\mathcal{G}}$ such that $X \cdot Y$ is well defined in $K(\mathcal{G})$, then $X \cdot Y \in K_{\mathcal{G}}$. Note that this implies that $H(X) \cdot H(Y)$ is a well defined element of $K_{\mathcal{L}}$ (cf. proof of Theorem 1.2 below).

Similarly we say that the quadruple has *duality*, if the pair $(\mathcal{G}, \mathcal{L})$ has duality and if furthermore for any element $X \in K_{\mathcal{G}}$ such that X^* is well defined as an element of $K(\mathcal{G})$, then $X^* \in K_{\mathcal{G}}$.

For a $\mathfrak{u}$-admissible pair, the quadruple $(\mathcal{G}, \mathcal{L}, K(\mathcal{G}), K(\mathcal{L}))$ is always $\mathfrak{u}$-admissible and it is multiplicative resp. has duality if the pair $(\mathcal{G}, \mathcal{L})$ is multiplicative resp. has duality.

For a $\mathfrak{u}$-admissible quadruple $\mathcal{Q}$ the localization

$$C_{\mathfrak{q}}(\mathcal{Q}) := K_{\mathcal{L}}[W_{\mathfrak{q}}^{-1}]$$

is well defined. It comes with a canonical group homomorphism $K_{\mathcal{L}} \to C_{\mathfrak{q}}(\mathcal{Q})$, which is injective in the finite-dimensional case, but not injective in general. Furthermore the partially defined multiplication of $K(\mathcal{L})$ has a natural (but still only partially defined) extension to $C_{\mathfrak{q}}(\mathcal{L})$, such that $W_{\mathfrak{q}}$ is invertible in $C_{\mathfrak{q}}(\mathcal{L})$ by the very definition of localization.

We define the *algebraic character* of $X \in K_{\mathcal{G}}$ with respect to $\mathfrak{q}$ as

$$c_{\mathfrak{q}}(X) := H(X)/W_{\mathfrak{q}} \in C_{\mathfrak{q}}(\mathcal{Q}).$$

The basic properties of $c_{\mathfrak{q}}$ are summarized in

Theorem 1.2. *The map $c_{\mathfrak{q}}$ is a group homomorphism $K_{\mathcal{G}} \to C_{\mathfrak{q}}(\mathcal{Q})$, carrying $[\mathbf{C}]$ to $[\mathbf{C}]$ (i.e. 1 to 1). If the restriction of $X \in K_{\mathcal{G}}$ to $(\mathfrak{l}, L \cap K)$ lies in $K_{\mathcal{L}}$ then as an element of $C_{\mathfrak{q}}(\mathcal{Q})$*

$$c_{\mathfrak{q}}(X) = [X].$$

If the quadruple is multiplicative and if for $X, Y \in K_{\mathcal{G}}$ the product $X \cdot Y$ lies in $K_{\mathcal{G}}$ as well, then

$$c_{\mathfrak{q}}(X \cdot Y) = c_{\mathfrak{q}}(X) \cdot c_{\mathfrak{q}}(Y).$$

If the quadruple has duality and if $X^* \in \mathcal{G}$, then

$$c_{\mathfrak{q}}(X^*) = c_{\mathfrak{q}}(X)^*.$$

Proof. Assume that V lies in $\mathcal{L}$. The $\mathfrak{u}$-cohomology of V may be computed from the finite-dimensional complex

$$\bigwedge^{\bullet} \mathfrak{u}^* \otimes_{\mathbf{C}} V,$$

which means that we have the Riemann-Roch formula

$$H(V) = \sum_q (-1)^q [\bigwedge^q \mathfrak{u}^*] \cdot [V]$$

in $K(\mathcal{L})$. This shows the first identity.

Assume now that $(\mathcal{G}, \mathcal{L})$ is multiplicative. Consider the diagonal embedding of Lie algebras

$$\Delta : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}, g \mapsto (g, g).$$

Furthermore we have two full categorical embeddings

$$i_{1,2} : \mathcal{G} \to \mathcal{G} \times \mathcal{G},$$

given by

$$i_1 : V \mapsto V \otimes_{\mathbf{C}} \mathbf{C},$$

$$i_2 : W \mapsto \mathbf{C} \otimes_{\mathbf{C}} W,$$

considered as $\mathfrak{g} \times \mathfrak{g}$-modules under the standard action on the tensor product

$$(g_1, g_2)(v \otimes w) = (g_1 v) \otimes w + v \otimes g_2 w.$$

By the Künneth Formula and the Hochschild-Serre spectral sequence for the diagonal empedding $\mathfrak{u} \to \mathfrak{u} \times \mathfrak{u}$, we have in $K(\mathcal{L})$

$$\sum_{p,q} (-1)^{p+q} [\bigwedge^p \mathfrak{u}^*] \cdot [H^q(\mathfrak{u}, V \otimes_{\mathbf{C}} W)] =$$

$$\sum_{p,q} (-1)^{p+q} [H^p(\mathfrak{u}, V)] \cdot [H^q(\mathfrak{u}, W)],$$

and consequently

$$W_{\mathfrak{q}} \cdot H(V \otimes_{\mathbf{C}} W) = H(V) \cdot H(W).$$

By the multiplicativity of $\mathcal{Q}$, the result follows as desired.

The duality statement follows from Poincaré duality, i.e. Corollary 3.8 of [9]. Let V be an object of $\mathcal{G}$. Strictly speaking we need to consider the $(\mathfrak{q}, L \cap K)$-dual of V instead of V^*, but in the case of finite length the resulting (co)homology is the same. This yields by the already proved multiplicativity

$$c_{\mathfrak{q}}(V^*) = (-1)^{\dim \mathfrak{u}} \cdot c_{\mathfrak{q}}(\bigwedge^{\dim \mathfrak{u}} \mathfrak{u})^* \cdot c_{\mathfrak{q}}(V)^*,$$

where we extended the definition of $c_{\mathfrak{q}}$ to the module $\bigwedge^{\dim \mathfrak{u}} \mathfrak{u}$. As the character of the trivial representation in the category of finite-dimensional representations is self-dual by the first statement of the theorem, we have

$$(-1)^{\dim \mathfrak{u}} \cdot c_{\mathfrak{q}}(\bigwedge^{\dim \mathfrak{u}} \mathfrak{u})^* = [\mathbf{C}],$$

concluding the argument. □

Proposition 1.3. *Let $\mathcal{Q}$ be a $\mathfrak{q}$-admissible quadruple, and assume that it satisfies the same admissibility conditions as above with cohomology H^q replaced by homology H_q. Then for any $X \in K_{\mathcal{G}}$*

$$c_{\mathfrak{q}}(X) = \frac{\sum_q (-1)^q [H_q(\mathfrak{u}, X)]}{\sum_q (-1)^q [H_q(\mathfrak{u}, \mathbf{C})]}.$$

Proof. This follows from hard duality [9, Corollary 3.8], combined with the analoguous argument used in the proof of the duality statement in Theorem 1.2. □

If $\mathfrak{q}$ is minimal, i.e. if $(\mathfrak{l}, L \cap K)$ is a Cartan subpair, we call $c_{\mathfrak{q}}$ the *absolute* character. Otherwise we say that $c_{\mathfrak{q}}$ is a *relative* character. This terminology is justified by

Proposition 1.4. *Let $(\mathfrak{p}, M \cap K) \subseteq (\mathfrak{q}, L \cap K)$ be germane parabolic pairs with nilpotent radicals $\mathfrak{n} \subseteq \mathfrak{u}$ respectively and let $(\mathcal{G}, \mathcal{L})$ resp. $(\mathcal{L}, \mathcal{M})$ be $\mathfrak{u}$- resp. $\mathfrak{n} \cap \mathfrak{l}$-admissible pairs. Assume that $(\mathcal{G}, \mathcal{M})$ is $\mathfrak{n}$-admissible and that the qaudruples $\mathcal{Q} = (\mathcal{G}, \mathcal{L}, K_{\mathcal{G}}, K_{\mathcal{L}})$, $\mathcal{Q}' = (\mathcal{L}, \mathcal{M}, K_{\mathcal{L}}, K_{\mathcal{M}})$ and $\mathcal{Q}'' = (\mathcal{G}, \mathcal{M}, K_{\mathcal{G}}, K_{\mathcal{M}})$ are $\mathfrak{u}$- resp. $\mathfrak{n} \cap \mathfrak{l}$- resp. $\mathfrak{n}$-admissible. Then*

$$c_{\mathfrak{p}} = c_{\mathfrak{p} \cap \mathfrak{l}} \circ c_{\mathfrak{q}}.$$

We remark that the first statement of Theorem 1.2 is a special case of this Proposition.

Here $c_{\mathfrak{p} \cap \mathfrak{l}}$ is considered as the character of an $(\mathfrak{l}, L \cap K)$-module with respect to the parabolic subpair induced by $(\mathfrak{p}, M \cap K)$, naturally extended to a map $C_{\mathfrak{q}}(\mathcal{Q}) \to C_{\mathfrak{p}}(\mathcal{Q}'')$.

Proof. Another application of the Hochschild-Serre spectral sequence similar to the proof of Proposition 1.5 below. □

The $\mathfrak{u}$-admissibility of $(\mathcal{G}, \mathcal{M})$ resp. $\mathcal{Q}''$ is a weak condition and is nearly automatic. We only need to assume that $\mathcal{M}$ contains the abutments of the Hochschild-Serre spectral sequences in question, and similarly for the Grothendieck groups. In this sense admissibility is transitive.

In principle this proposition reduces the study of characters of $(\mathfrak{g}, K)$-modules to the study of relative characters for maximal parabolic subalgebras.

Let $\iota : (\mathfrak{g}', K') \to (\mathfrak{g}, K)$ be an inclusion of reductive pairs, so in particular it is assumed to be compatible with the Cartan involutions and all the other data associated to the pairs. We assume that we are given categories $\mathcal{G}'$ resp. $\mathcal{G}$ such that forgetting along ι induces a well defined functor $\mathcal{G} \to \mathcal{G}'$.

Choose a Cartan subpair $(\mathfrak{h}', H' \cap K')$ in $(\mathfrak{g}', K')$. We assume furthermore that there is an element $h \in \mathfrak{h}'$ that is regular in $\mathfrak{g}$ and such that all roots of $\mathfrak{g}$ on h are real (cf. [9, Proposition 4.70]). Let $\mathfrak{q}$ resp. $\mathfrak{q}'$ be the associated germane parabolic subalgebras given by the non-negative eigenspaces of $\mathrm{ad}(h)$, L and L' the respective Levi factors. Then $\mathfrak{q} \cap \mathfrak{g}' = \mathfrak{q}'$ and in particular $L' \cap K' = L \cap K'$. Write $\mathfrak{l}$ resp. $\mathfrak{l}'$ for the complexified Lie algebras of L and L' and $\mathfrak{u}$ resp. $\mathfrak{u}'$ for the nilpotent radicals of $\mathfrak{q}$ resp. $\mathfrak{q}'$.

Fix two categories $\mathcal{L}$ and $\mathcal{L}'$ such that $(\mathcal{G}, \mathcal{L})$ and $(\mathcal{G}', \mathcal{L}')$ are $\mathfrak{u}$- resp. $\mathfrak{u}'$-admissible and restriction along ι again gives rise to a well defined functor $\iota : \mathcal{L} \to \mathcal{L}'$.

Let $Q = (\mathcal{G}, \mathcal{L}, K_{\mathcal{G}}, K_{\mathcal{L}})$ be $\mathfrak{u}$- and $Q' = (\mathcal{G}', \mathcal{L}', K_{\mathcal{G}'}, K_{\mathcal{L}'})$ be $\mathfrak{u}'$-admissible. We assume that ι induces well defined maps

$$\iota : K_{\mathcal{G}} \to K_{\mathcal{G}'}$$

and

$$\iota : K_{\mathcal{L}} \to K_{\mathcal{L}'}.$$

We define

$$W_{\mathfrak{q}/\mathfrak{q}'} := W_{\mathfrak{q}}|_{\mathfrak{l}', L' \cap K'} / W_{\mathfrak{q}'},$$

and assume that it is an element of $K_{\mathcal{L}'}$, and that the latter is closed under tensorization with $W_{\mathfrak{q}/\mathfrak{q}'}$. Then the homorphism

$$\iota : C_{\mathfrak{q}}(\mathcal{Q}) \to C_{\mathfrak{q}'}(\mathcal{Q}')[W_{\mathfrak{q}/\mathfrak{q}'}^{-1}],\ c \mapsto c|_{\mathfrak{l}',L'\cap K'}$$

is well defined. We denote the corresponding homomorphism $K_{\mathcal{G}} \to K_{\mathcal{G}'}$ defined by restriction along ι the same.

Proposition 1.5. *Let $\iota : (\mathfrak{g}', K') \to (\mathfrak{g}, K)$ be an inclusion of reductive pairs as above, in particular mapping one regular element to a regular one. Assume furthermore that the restriction along ι is compatible with the quadruples $\mathcal{Q}$ and $\mathcal{Q}'$ as above.*

Then restriction defines an additive and multiplicative map $\iota : C_{\mathfrak{q}}(\mathcal{Q}) \to C_{\mathfrak{q}'}(\mathcal{Q}')[W_{\mathfrak{q}/\mathfrak{q}'}^{-1}]$, respecting duals, with the property

$$\iota \circ c_{\mathfrak{q}} = c_{\mathfrak{q}'} \circ \iota.$$

Proof. We have a Hochschild-Serre spectral sequence

$$E_1^{p,q} = \bigwedge^p (\mathfrak{u}/\mathfrak{u}')^* \otimes_{\mathbf{C}} H^q(\mathfrak{u}', X'),$$

$$E_1^{p,q} \Longrightarrow H^{p+q}(\mathfrak{u}, X).$$

All differentials in this spectral sequence are indeed $(\mathfrak{l}', L' \cap K')$-linear. On the one hand, by our assumption on ι the cohomology $H^q(\mathfrak{u}', X')$ lies in $\mathcal{L}'$, as does $H^{p+q}(\mathfrak{u}, X)$ and hence we deduce in $K(\mathcal{L}')$ the identity

$$\sum_p (-1)^p \cdot [\bigwedge^p (\mathfrak{u}/\mathfrak{u}')^*] \cdot \sum_q (-1)^q \cdot [H^q(\mathfrak{u}', X)] = \sum_p (-1)^p \cdot [H^p(\mathfrak{u}, X)]|_{\mathfrak{l}',L'\cap K'}.$$

On the other hand, due to our hypothesis

$$\sum_p (-1)^p \cdot [\bigwedge^p (\mathfrak{u}/\mathfrak{u}')^*] = W_{\mathfrak{q}/\mathfrak{q}'}.$$

Obviously ι is multiplicative and respects duals, concluding the proof. $\square$

An immediate application to admissible modules is the comparison between the *full* and the *compact character*, i.e. the character of an admissible $(\mathfrak{g}, K)$-module X with respect to a maximally compact L and θ-stable parabolic $\mathfrak{q}$ to its restriction to K or even K^0. Both cases are covered by Proposition 1.5. This shows the relation between characters and generalized Blattner formulas. It is an easy excercise to deduce the Blattner formulas formally from the known character formulas. We come back to this question in the last section, after studying localizations in more detail.

As we have a purely algebraic proof of the Blattner formula for the Zuckerman-Vogan modules $A_{\mathfrak{q}}(\lambda)$, due to Zuckerman, it would be interesting to deduce from this the character formulas. At least for the discrete series this actually works, and in some sense, inverts the approach taken by Hecht and Schmid in their proof of the Blattner conjecture for the discrete series [5].

2 Translation functors

By the theory of the Jantzen-Zuckerman translation functors representations occur in 'coherent families'. As Vogan has shown ([21], [9, Theorem 7.242]), $\mathfrak{u}$-cohomology behaves well under translation functors, which implies that algebraic characters do so as well. We make this precise in this section.

We use the following notation. We do not assume that K is connected nor that $\mathfrak{q}$ is minimal, but we assume it to be θ-stable. If λ is a character of $\mathfrak{h}$, we write $\mathcal{P}_\lambda$ for the endofunctor $\mathcal{C}_{\mathrm{fl}}(\mathfrak{g},K) \to \mathcal{C}_{\mathrm{fl}}(\mathfrak{g},K)$, which projects to the λ-primary component for the action of $Z(\mathfrak{g})$. This functor extends to a functor

$$\mathcal{P}_\lambda : \mathcal{C}_{\mathrm{fd}}(\mathfrak{g},K) \to \mathcal{C}_{\mathrm{fl}}(\mathfrak{g},K).$$

By Proposition 7.20 of loc. cit., $\mathcal{P}_\lambda$ is exact and hence induces a map on the Grothendieck groups. Let λ' be another character of $\mathfrak{h}$ such that $\mu := \lambda' - \lambda$ is algebraically integral. Write F^μ for an irreducible finite-dimensional $(\mathfrak{g},K)$-representation of extreme weight μ which remains irreducible as a $\mathfrak{g}$-module (and is supposed to exist). Then one can define the *translation functor*

$$\psi_{\lambda,F^\mu}^{\lambda'} := \mathcal{P}_{\lambda'} \circ (\,\cdot \otimes_{\mathbf{C}} F^\mu) \circ \mathcal{P}_\lambda.$$

This is an exact functor $\mathcal{C}_{\mathrm{fd}}(\mathfrak{g},K) \to \mathcal{C}_{\mathrm{fl}}(\mathfrak{g},K)$ that descends to the Grothendieck group as well. In our terminology Theorem 7.242 of loc. cit. reads as follows. Write E^μ for the irreducible $(\mathfrak{l}, L\cap K)$-submodule of F^μ containing the weight space for μ.

Suppose that λ and μ satisfy

(i) $\lambda + \rho(\mathfrak{u})$ and $\lambda + \mu + \rho(\mathfrak{u})$ are integrally dominant relative to $\Delta^+(\mathfrak{g},\mathfrak{h})$,

(ii) K fixes the $Z(\mathfrak{g})$ infinitesimal characters $\chi_{\lambda+\rho(\mathfrak{u})}$ and $\chi_{\lambda+\mu+\rho(\mathfrak{u})}$,

(iii) $L\cap K$ fixes the $Z(\mathfrak{l})$ infinitesimal characters χ_λ and $\chi_{\lambda+\mu}$,

(iv) $\lambda + \rho(\mathfrak{u})$ is at least as singular as $\lambda + \mu + \rho(\mathfrak{u})$.

Then for any $(\mathfrak{g},K)$-module X of finite length, and as a consequence also for any module in $\mathcal{C}_{\mathrm{fd}}(\mathfrak{g},K)$,

$$\psi_{\lambda,E^\mu}^{\lambda+\mu}(W_\mathfrak{q}\cdot c_\mathfrak{q}(X)) = \mathcal{P}_{\lambda+\mu}(W_\mathfrak{q}\cdot c_\mathfrak{q}(\psi_{\lambda+\rho(\mathfrak{u}),F^\mu}^{\lambda+\mu+\rho(\mathfrak{u})}(X))).$$

Note that on the one hand, the proof of Theorem 7.242 eventually shows more. Namely that

$$\psi_{w(\lambda+\rho(\mathfrak{u}))-\rho(\mathfrak{u}),E^{w(\mu)}}^{w(\lambda+\mu+\rho(\mathfrak{u}))-\rho(\mathfrak{u})}(W_\mathfrak{q}\cdot c_\mathfrak{q}(X)) = \mathcal{P}_{w(\lambda+\mu+\rho(\mathfrak{u}))-\rho(\mathfrak{u})}(W_\mathfrak{q}\cdot c_\mathfrak{q}(\psi_{\lambda+\rho(\mathfrak{u}),F^\mu}^{\lambda+\mu+\rho(\mathfrak{u})}(X))) \qquad (4)$$

for any $w\in W(\mathfrak{g},\mathfrak{h})$.

Assume now that K is connected and $\mathfrak{l} = \mathfrak{h}$, i.e. $\mathfrak{q}$ is minimal. Then E_μ is always one-dimensional. Assume furthermore that X has infinitesimal character

$\lambda + \rho(\mathfrak{u})$. Then the parameter λ is essentially translated by μ (the projections have no effect in this case). Furthermore F^μ exists always and conditions (ii) and (iii) become vacuous as K is connected. To study the effect on characters we make use of Proposition 0.6 which says that

$$\mathcal{P}_\nu(W_\mathfrak{q} \cdot c_\mathfrak{q}(X)) \neq 0$$

implies that $\nu = w(\lambda + \rho(\mathfrak{u})) - \rho(\mathfrak{u})$ for some $w \in W(\mathfrak{g}, \mathfrak{h})$. Applying the twist E^μ and another projection we arrive at the question when

$$\mathcal{P}_\nu(\mathcal{P}_{w(\lambda+\rho(\mathfrak{u}))-\rho(\mathfrak{u})}(W_\mathfrak{q} \cdot c_\mathfrak{q}(X)) \cdot [E^\mu]) \neq 0.$$

This amounts to $\nu = w(\lambda+\rho(\mathfrak{u}))+\mu-\rho(\mathfrak{u})$. Consequently we can detect all multiplicities of non-zero $U(\mathfrak{l})$-isotypic components in $W_\mathfrak{q} \cdot c_\mathfrak{q}(X)$ by consideration of

$$\psi^{w(\lambda+\mu+\rho(\mathfrak{u}))-\rho(\mathfrak{u})}_{w(\lambda+\rho(\mathfrak{u}))-\rho(\mathfrak{u}),E^{w(\mu)}}(W_\mathfrak{q} \cdot c_\mathfrak{q}(X)),$$

for all $w \in W(\mathfrak{g}, \mathfrak{h})$.

Now we know by Proposition 0.6 that

$$W_\mathfrak{q} \cdot c_\mathfrak{q}(X) = \sum_{w \in W(\mathfrak{g},\mathfrak{h})} n_w^\lambda [\mathbf{C}_{w(\lambda+\rho(\mathfrak{u}))-\rho(\mathfrak{u})}] \tag{5}$$

for integers $n_w^\lambda \in \mathbf{Z}$ and similarly for $\psi^{\lambda+\mu}_{\lambda,F^\mu}(X)$. Note that the coefficients are not uniquely determined in general. Plugging all this together into (4) we find

Proposition 2.1. *Under the above hypothesis we may assume in* (5) *that*

$$n_w^\lambda = n_w^{\lambda+\mu}$$

for all $w \in W(\mathfrak{g}, \mathfrak{h})$.

3 Applications

Compact groups resp. finite-dimensional representations

We consider the special case where G is compact connected, $(\mathfrak{l}, L)$ is a Cartan subpair and the $\mathfrak{u}$-admissible pair $(\mathcal{G}, \mathcal{L})$ consists of the categories of finite-dimensional representations (which are just the representations of finite length). We set $K_\mathcal{G} := K(\mathcal{G})$ and $K_\mathcal{L} := K(\mathcal{L})$ and consider the corresponding $\mathfrak{u}$-admissible quadruple $\mathcal{Q}$. Note that in this special situation the canonical map $K_\mathcal{L} \to C_\mathfrak{q}(\mathcal{Q})$ is injective. Theorem 1.2 then says that

$$c_\mathfrak{q} : K(\mathcal{G}) \to C_\mathfrak{q}(\mathcal{Q})$$

is a ring homomorphism which factors over the forgetful map

$$K(\mathcal{G}) \to K(\mathcal{L}).$$

In particular algebraic characters characterize finite-dimensional representations up to isomorphism by the classical highest weight theory. Furthermore Kostant's Theorem gives

Theorem 3.1 (Weyl Character Formula)**.** *If G is compact connected and V is irreducible of highest weight λ, then we have*

$$c_{\mathfrak{q}}(V) = \frac{\sum_{w\in W(\mathfrak{g},\mathfrak{l})}(-1)^{\ell(w)}[\mathbf{C}_{w(\lambda+\rho(\mathfrak{u}))-\rho(\mathfrak{u})}]}{\sum_{w\in W(\mathfrak{g},\mathfrak{l})}(-1)^{\ell(w)}[\mathbf{C}_{w(\rho(\mathfrak{u}))-\rho(\mathfrak{u})}]}.$$

The standard complex for $\mathfrak{u}$-cohomology resp. shows

Proposition 3.2 (Weyl Denominator Formula)**.**

$$W_{\mathfrak{q}} = \sum_{w\in W(\mathfrak{g},\mathfrak{l})}(-1)^{\ell(w)}[\mathbf{C}_{w(\rho(\mathfrak{u}))-\rho(\mathfrak{u})}] = \prod_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{l})}(1-[\mathbf{C}_{-\alpha}]).$$

Modules of finite length

In this section $(\mathfrak{g},K)$ is a reductive pair associated to a linear[1] real reductive Lie group in Harish-Chandra's class and we keep the rest of the notation as before. We assume that the pair of $\mathfrak{u}$-admissible categories $(\mathcal{G},\mathcal{L})$ is given by the pair

$$(\mathcal{C}_{\mathrm{fl}}(\mathfrak{g},K),\mathcal{C}_{\mathrm{fl}}(\mathfrak{l},L\cap K))$$

of finite length modules, and with $K_{\mathcal{G}} = K(\mathcal{G})$ and $K_{\mathcal{L}} = K(\mathcal{L})$ we get a $\mathfrak{u}$-admissible quadruple $\mathcal{Q}$. All identities are to be understood in $C_{\mathfrak{q}}(\mathcal{Q})$.

Proposition 3.3. *Let X be a $(\mathfrak{g},K)$-module of finite length. Then the numerator in the character $c_{\mathfrak{q}}(X)$ is of finite length (i.e. a finite linear combination of finite length modules). If X has infinitesimal character χ and if $\mathfrak{q}$ is minimal, then there exist integers $n_w\in\mathbf{Z}$ for $w\in W(\mathfrak{g},\mathfrak{l})$ such that*

$$W_{\mathfrak{q}}\cdot c_{\mathfrak{q}}(X) = \sum_{w\in W(\mathfrak{g},\mathfrak{l})} n_w[\mathbf{C}_{w(\chi)-\rho(\mathfrak{u})}]$$

Proof. This is a mere restatement of Proposition 0.6 and Corollary 0.8 in this setting. □

Theorem 3.4. *Let V,W be two $(\mathfrak{g},K)$-modules of finite length and assume that*

$$c_{\mathfrak{q}}(V) = c_{\mathfrak{q}}(W)$$

for every minimal germane parabolic subalgebra $\mathfrak{q}$. Then V has the same composition factors with the same multiplicities as W (i.e. V and W have the same semi-simplifications).

Proof. It will turn out in the proof that if $V\neq 0$, then $H_{\mathfrak{q}}(V)\in C_{\mathfrak{q}}(\mathcal{Q})$ does not vanish for all $\mathfrak{q}$. Therefore we may assume that K is connected, as the action of the component group is reflected by the action of $L\cap K$ on cohomology.

By Matsuki [18] every G-conjugacy class of minimal parabolic subalgebras contains a germane parabolic subalgebra $\mathfrak{q}$. If $\mathfrak{q}$ is real, Osborne's Conjecture

[1] Linearity is only included for some minor technical reasons in [21].

[19], as proven by Hecht and Schmid [6], tells us that the restriction of Harish-Chandra's global character $\Theta(V)$ to a 'big' open subset U of L, i.e. whose conjugates cover L, coincides with Harish-Chandra's global character $\Theta(c_{\mathfrak{q}}(V))$, which is formally associated to $c_{\mathfrak{q}}(V)$, restricted to the same set U. If $\mathfrak{q}$ is θ-stable, Vogan has shown that the analoguous statement is true without restricting to a subset of L [21, Theorem 8.1]. The general case may be reduced to these two cases as follows.

As before, $\mathfrak{q}$ gives rise to a θ-stable Levi pair $(\mathfrak{l}, L\cap K)$, which is a Cartan pair in our setting. We then have a real form $\mathfrak{h}_0\subseteq\mathfrak{g}_0$ of $\mathfrak{h}$ and decompose it into $\mathfrak{l}_0=\mathfrak{t}_0\oplus\mathfrak{a}_0$, where $\mathfrak{t}_0=\mathfrak{k}_0\cap\mathfrak{l}_0$ and $\mathfrak{a}_0=\mathfrak{l}_0\cap\mathfrak{p}_0$, where $\mathfrak{p}_0$ is the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$. Now choose an ordering of the non-zero weights of $\mathfrak{t}$ occuring in $\mathfrak{g}$, which gives rise to a subset $\Delta^+(\mathfrak{g},\mathfrak{t})\subseteq\Delta(\mathfrak{g},\mathfrak{t})$ of the set of non-zero roots of $\mathfrak{t}$ occuring in $\mathfrak{g}$. From this we deduce a subset $\Delta^+\subseteq\Delta(\mathfrak{g},\mathfrak{l})$ in the following way. A root α is in Δ^+ if and only if

$$\alpha|_{\mathfrak{t}}\in\Delta^+(\mathfrak{g},\mathfrak{t})\cup\{0\}.$$

Then Δ^+ contains some positive system $\Delta^+(\mathfrak{g},\mathfrak{l})$, is closed under addition in $\Delta(\mathfrak{g},\mathfrak{h})$, and is θ-stable by definition. Hence it defines a θ-stable parabolic pair $(\mathfrak{q}', L'\cap K)$ with a Levi pair $(\mathfrak{l}', L'\cap K)$ containing the Cartan pair $(\mathfrak{l}, L\cap K)$. By the result of Vogan loc. cit. the identity

$$c_{\mathfrak{q}'}(V)=c_{\mathfrak{q}'}(W)$$

implies that the restriction of the corresponding characters of Harish-Chandra coincide on L. Hence by Harish-Chandra's classical results [2, 3, 4] combined with Proposition 1.4 this reduces the problem to $(\mathfrak{g},K)=(\mathfrak{l}',L'\cap K)$.

Now all roots of $\mathfrak{l}$ in $\mathfrak{l}'$ are real by construction, as for any $\alpha\in\Delta(\mathfrak{l}',\mathfrak{l})$

$$\alpha|_{\mathfrak{t}}=0.$$

Hence we are in a position where we can appeal to the aforementioned result of Hecht and Schmid, proving the claim of the theorem again thanks to Harish-Chandra's work. $\square$

4 Algebraic characters and Blattner formulas

In this section we use results about the kernels of the localization maps to establish an explicit relation between character formulas and generalized Blattner formulas. The results about localization maps will be established in the last section.

We fix a reductive pair $(\mathfrak{g},K)$ and assume that K is connected for simplicity. Furthermore fix a θ-stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}\subseteq\mathfrak{g}$, containing a Borel subalgebra $\mathfrak{p}=\mathfrak{t}+\mathfrak{n}$ of $\mathfrak{k}$, where $\mathfrak{t}\subseteq\mathfrak{l}$ is the complexified Lie algebra of a maximal torus $T\subseteq L\cap K$ and $\mathfrak{n}$ is the nilpotent radical. We denote $X(T)$ the group of characters of T. On $X(T)$ choose the ordering for which the weights occuring in $\mathfrak{p}$ are non-negative.

Choose an abelian category $\mathcal{G}$ of $(\mathfrak{g}, K)$-modules and fix an arbitrary weight $\lambda_0 \in X(T)$ and define the full subcategory $\mathcal{G}_{\lambda_0}^{(\mathfrak{q})} \subseteq \mathcal{G}$ consisting of all modules X in $\mathcal{G}$ with the following property:

(S) For any $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$ and any highest weight λ of a K-type occuring in X we have for any $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$ and any $w \in W(K,T) = W(\mathfrak{k}, \mathfrak{t})$

$$|\langle w(\lambda + \rho(\mathfrak{n})) - \lambda_0, \alpha\rangle| \;\geq\; \left(\dim \mathfrak{n} + \frac{1}{2}\right) \cdot \langle \alpha, \alpha\rangle + \dim \mathfrak{n} \cdot \sum_{\substack{\beta \in \Delta(\mathfrak{u},\mathfrak{t})\\ \beta \neq \alpha}} \langle \alpha, \beta\rangle. \quad (6)$$

Then $\mathcal{G}_{\lambda_0}^{(\mathfrak{q})}$ is abelian and if $(\mathcal{G}, \mathcal{L})$ is a $\mathfrak{q}$-admissible pair, then so is $(\mathcal{G}_{\lambda_0}^{(\mathfrak{q})}, \mathcal{L})$. If $(\mathcal{G}, \mathcal{L})$ is multiplicative or has duality, then $(\mathcal{G}_{\lambda_0}^{(\mathfrak{q})}, \mathcal{L})$ has the same property.

Let us write $\mathcal{K}_{\lambda_0}^{(\mathfrak{q})}$ for the category of $(\mathfrak{k}, K)$-modules that arises when restricting objects of $\mathcal{G}_{\lambda_0}^{(\mathfrak{q})}$ to $(\mathfrak{k}, K)$. It comes with a surjective faithful functor $\iota : \mathcal{G}_{\lambda_0}^{(\mathfrak{q})} \to \mathcal{K}_{\lambda_0}^{(\mathfrak{q})}$.

Assume that $\mathcal{G}_{\lambda_0}^{(\mathfrak{q})}$ resp. $\mathcal{L}$ are subcategories of $\mathcal{C}_{\mathrm{f}}(\mathfrak{g}, K)$ resp. $\mathcal{C}_{\mathrm{f}}(\mathfrak{l}, L \cap K)$, and write $\mathcal{T}$ for $\mathcal{C}_{\mathrm{f}}(\mathfrak{t}, T)$. We may assume that objects from $\mathcal{L}$ restrict to objects in $\mathcal{T}$, and that $(\mathcal{G}_{\lambda_0}^{(\mathfrak{q})}, \mathcal{L})$ is $\mathfrak{q}$-admissible.

Then the pair $(\mathcal{K}_{\lambda_0}^{(\mathfrak{q})}, \mathcal{T})$ is $\mathfrak{p}$-admissible and we have the diagram

$$\begin{tikzcd}
K(\mathcal{G}_{\lambda_0}^{(\mathfrak{q})}) \arrow[r, "\iota"] \arrow[d, "c_{\mathfrak{q}}"'] & K(\mathcal{K}_{\lambda_0}^{(\mathfrak{q})}) \arrow[d, "c_{\mathfrak{p}}"'] \\
C_{\mathfrak{q}}(\mathcal{L}) \arrow[r, "\iota"] & C_{\mathfrak{p}}(\mathcal{T})[W_{\mathfrak{q}/\mathfrak{p}}^{-1}]
\end{tikzcd}$$

which is commutative by Proposition 1.5. The goal of this section is to show

Theorem 4.1. *For any X in $\mathcal{G}_{\lambda_0}^{(\mathfrak{q})}$ its restriction $\iota(X)$ to K is uniquely determined by*

$$\iota(c_{\mathfrak{q}}(X)) \in C_{\mathfrak{p}}(\mathcal{T})[W_{\mathfrak{q}/\mathfrak{p}}^{-1}]$$

as a preimage under the map

$$c_{\mathfrak{p}} : K(\mathcal{K}_{\lambda_0}^{(\mathfrak{q})}) \to C_{\mathfrak{p}}(\mathcal{T})[W_{\mathfrak{q}/\mathfrak{p}}^{-1}].$$

The idea of proof is to compute the kernel of the map

$$c_{\mathfrak{p}} : K(\mathcal{C}_{\mathrm{a}}(\mathfrak{k}, K)) \to C_{\mathfrak{p}}(\mathcal{C}_{\mathrm{a}}(\mathfrak{t}, T))[W_{\mathfrak{q}/\mathfrak{p}}^{-1}].$$

The statement of Theorem 4.1 then is equivalent to saying that this kernel intersects $K(\mathcal{K}_{\lambda_0}^{(\mathfrak{q})})$ only trivially. We remark that knowing the kernel implicitly also gives information about the cases violating condition (6).

It is crucial for our proof relies on the fact that the multiplicities of K-types in finite length modules are asymptotically bounded by the square of their dimensions, by a theorem of Harish-Chandra, which at the same time is a necessary

condition for the global character to exist. However the algebraic picture allows us in principle to relax this boundedness, and to raise the exponent, i.e. only require that the multiplicities are asymptotically bounded by (a constant multiple of) the d-th power of their dimensions, where d is fixed for the category of admissible $(\mathfrak{g}, K)$-modules in question. The choice of d then determines the lower bound in the regularity condition (6). As we will see in Section 5 below, the bound, i.e. the right hand side of (6) behaves linearly in d.

Finally we point out that our method below is universal to any branching problem related to restrictions of reductive pairs in the sense of Proposition 1.5 in the context where the character on the smaller group is absolute. It would be desirable to generalize this approach to the relative case, which in turn may be reduced to the case of a maximal parabolic, again by Proposition 1.5.

Proof. First we observe that Harish-Chandra's bound on the multiplicity of the K-types in an irreducible K-type X together with the Weyl dimension formula shows that the multiplicities m_λ of K-types in X with highest weight λ are bounded by

$$m_\lambda \;\leq\; C \cdot \prod_{\beta\in\Delta(\mathfrak{n},\mathfrak{t})} \langle \lambda + \rho(\mathfrak{n}), \beta\rangle^2,$$

for some constant $C > 0$ depending on X. In the sequel we use the notation and terminology from section 6. Then if the series

$$m := \sum_\lambda m_\lambda \cdot \lambda \in \mathbf{C}[[\Lambda]]$$

lies primitively in the kernel of some $t_{\underline{\alpha}}^{\underline{n}}$, we see with Corollary 6.7 and Proposition 6.8, imposing temporarily that $\lambda_0 = 0$, that m_λ is a $\mathbf{C}$-linear combination of products of the terms

$$\binom{n_i - 1 + k_i}{n_i - 1}, \quad \frac{n_i - 1 + 2k_i}{n_i - 1 + k_i} \cdot \binom{n_i - 1 + k_i}{n_i - 1},$$

where $1 \leq i \leq r$ and $0 \leq k_i \in \mathbf{Z}$, subject to

$$\frac{\langle \lambda, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} - \frac{n_i + 1}{2} \;<\; k_i \;\leq\; \frac{\langle \lambda, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} - \frac{n_i - 1}{2}.$$

We assume for simplicity that

$$k_i = \frac{\langle \lambda, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} - \frac{n_i - 1}{2},$$

which is legitimate when it comes to questions about asymptotic behavior. In the same spirit we may ignore the second term stemming from multiplication with $d_{\alpha_i,+}$, as in a single product they do not occur simultaneously. Then we may think of m_λ as a linear combination of products of the terms

$$\binom{\frac{n_i-1}{2} + \frac{\langle \lambda, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}}{n_i - 1},$$

each being a polynomial of degree $n_i - 1$ in $\langle \lambda, \alpha_i \rangle$.

On the one hand m is primitive in our setup, which by Corollary 6.7 implies that we may assume $\langle \lambda, \alpha_i \rangle$ arbitrarily large, and furthermore that the monomial

$$\prod_{i=1}^{r} \langle \lambda, \alpha_i \rangle^{n_i}$$

occurs in m_λ.

On the other hand we know that m_λ is bounded as above, which implies that

$$\sum_{i=1}^{r} n_i \ \leq \ 2 \dim \mathfrak{n}. \tag{7}$$

This in particular shows that we only need to consider the finitely many cases satisfying (7). In particular we may assume that $n_1, \ldots, n_r \leq 2 \dim \mathfrak{n}$, which in turn implies the same statement for arbitrary λ_0.

This shows that the regularity condition (6) then implies condition (16), which in turn guarantees that the kernel of the map $c_\mathfrak{p}$ intersects $K(\mathcal{K}_{\lambda_0}^{(\mathfrak{q})})$ only trivially by Kostant's Theorem. □

Consider the Blattner formula problem for $(\mathfrak{gl}_3, \mathrm{SO}(3))$, which leads to the non-reduced rank 1 root system $\pm\alpha, \pm 2\alpha$. We write Z_λ for an irreducible representation of SO(3) with hightest weight λ. Using the explicit description of the kernel of the localization map given in section 6, we deduce that in this case the kernel of the map

$$c_\mathfrak{p} : K(\mathrm{SO}(3)) \ \to \ C_\mathfrak{p}(\mathcal{T})[W_{\mathfrak{q}/\mathfrak{p}}^{-1}],$$

where $K(\mathrm{SO}(3))$ is the subgroup of $K(\mathcal{C}_\mathrm{a}(\mathfrak{so}_3, \mathrm{SO}(3)))$ satisfying Harish-Chandra's bound, is generated by the four elements

$$\kappa_1 := \sum_{k=0}^{\infty} (1 + 2k) \cdot [Z_{k\alpha}],$$

$$\kappa_2 := \sum_{k=0}^{\infty} (1 + 2k) \cdot [Z_{(1+2k)\alpha}],$$

$$\kappa_3 := \sum_{k=0}^{\infty} 2k \cdot [Z_{2k\alpha}],$$

$$\kappa_4 := \sum_{k,l=0}^{\infty} (1 + 2k)(1 + l) \cdot [Z_{(2+k+2l)\alpha}],$$

which correspond to $d_{\alpha,+} \cdot y_{(\alpha)}^{(2)}$, $d_{\alpha,+} \cdot y_{(2\alpha)}^{(2)}$, $d_{\alpha,+}^{(3)} \cdot y_{(\alpha)}^{(2)}$, and $d_{\alpha,+} \cdot y_{(\alpha,2\alpha)}^{(2,2)}$ respectively. These elements are linearly independent in $K(\mathrm{SO}(3)) \otimes_\mathbf{Z} \mathbf{Q}$ and each of them contains a SO(3)-type $Z_{a \cdot \alpha}$ with $0 \leq a \leq 2$.

It is easy to see that there is no integral linear combination of these elements such that the resulting (virtual) representation has minimal SO(3)-type Z_3 with multiplicity one, yet the projection to the first 4 components of these elements is still linearly independent. We conclude that whenever Z is an irreducible $(\mathfrak{gl}_3, \mathrm{SO}(3))$-module with vanishing $\iota(c_{\mathfrak{u}}(Z))$, then Z contains one of the SO(3)-types $Z_{a\cdot\alpha}$ with $0 \leq a \leq 2$, and the Blattner formula is a linear combination of $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, subject to the condition that the minimal SO(3)-type occurs with multiplicity one.

In higher rank cases the kernel is usually infinitely generated, yet there are certain restrictions when it comes to non-virtual representations, which are related to the action of the Weyl group of K. We may project onto the $(-1)^{\ell(\cdot)}$-isotypic subspace of this action, and the result is the kernel of the localization map

$$K(\mathcal{T}) \to C_{\mathfrak{p}}(\mathcal{T})[W_{\mathfrak{q}/\mathfrak{p}}^{-1}]$$

coming from virtual representations of K, which is obivously infinitely generated, even in the rank 1 case, if we do not impose Harish-Chandra's bound. After imposing it in the higher rank setting, there might still be possibly infinitely many generators left. This stems from the fact that whenever an element y lies in the kernel $t_{\underline{\alpha}}^{\underline{n}}$, there might be a weight β orthogonal to all the roots α_i occuring in α, and in particular it may be orthogonal to all the weights on which y is supported. Consequently any Laurent series f in β yields new elements $f \cdot y$, the latter being well defined. At the same time multiplication with a power of β may have the effect that even if the Weyl orbit of y corresponded to a non-virtual representation, the orbit of $f \cdot y$ may not (or vice versa), as in the corresponding same Weyl orbits opposit signs may occur simultaneously. For the root system of type A_2 and its non-reduced extensions the monoid of true representations in the kernel seems to be finitely generated.

5 Discretely decomposable modules

In this section we generalize the results from the previous section to discretely decomposable modules, again postponing the treatment of localizations to the section 6. We assume that we are in the setting as in Proposition 1.5, and we use the notation as introduced there, subject to the following restrictions.

We assume that the parabolic $\mathfrak{q}'$ is minimal, and that the $\mathfrak{u}'$-admissible categories are given by the pair

$$(\mathcal{G}', \mathcal{L}') = (\mathcal{C}_{\mathrm{fd}(b)}(\mathfrak{g}', K'), \mathcal{C}_{\mathrm{fd}}(\mathfrak{l}', L' \cap K'))$$

of discretely decomposable modules with finite multiplicities bounded by $b \geq 0$, i.e. $\mathcal{C}_{\mathrm{fd}}^b(\mathfrak{g}', K')$ denotes the full subcategory of $\mathcal{C}_{\mathrm{fd}}(\mathfrak{g}', K')$ consisting of modules X with the property that there are constants $c_X, d_X \geq 0$ such that for any irreducible Z with infinitesimal character λ its multiplicity satisfies

$$m_Z(X) \ \leq \ c_X \cdot \|\lambda\|^b + d_X, \tag{8}$$

the norm being any fixed euclidean norm on $X(\mathfrak{l}')$.

On the side of $(\mathfrak{g}, K)$ and $(\mathfrak{l}, L \cap K)$ we suppose that we are given $\mathfrak{u}$-admissible categories $(\mathcal{G}, \mathcal{L})$ (for example modules of finite length as $\mathcal{G}$) with the property that their restrictions are in $\mathcal{G}'$ resp. $\mathcal{L}'$.

Now fix a character $\lambda_0 \in X(\mathfrak{l}')$ and define the full subcategory $\mathcal{G}_{\lambda_0}^{(\mathfrak{q},b)} \subseteq \mathcal{G}$ consisting of all modules X in $\mathcal{G}$ with the following property:

(S) For any $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$ and any irreducible Z-type with infinitesimal character λ occuring in the restriction of X to $(\mathfrak{g}', \mathfrak{l}')$ we have for any $\alpha \in \Delta(\mathfrak{u}, \mathfrak{l}')$ and any $w \in W(\mathfrak{g}', \mathfrak{l}')$

$$|\langle w(\lambda) - \lambda_0, \alpha \rangle| \;\geq\; \frac{b+1}{2} \cdot \langle \alpha, \beta \rangle + \frac{b}{2} \cdot \sum_{\substack{\beta \in \Delta(\mathfrak{u}, \mathfrak{l}') \\ \beta \neq \alpha}} \langle \alpha, \beta \rangle, \tag{9}$$

And we define the $\mathfrak{u}$-admissible quadruple $\mathcal{Q}$ as in the previous section, in particular $\mathcal{G}'^{(\mathfrak{q},b)}_{\lambda_0}$ is the essential image of $\mathcal{G}_{\lambda_0}^{(\mathfrak{q},b)} \subseteq \mathcal{G}$ under the restriction map.

Theorem 5.1. *For any X in $\mathcal{G}_{\lambda_0}^{(\mathfrak{q},b)}$ the $\mathfrak{q}'$-character of any composition factor Z of its restriction $\iota(X)$ to $(\mathfrak{g}', K')$ is uniquely determined by*

$$\iota(c_{\mathfrak{q}}(X)) \in C_{\mathfrak{q}'}(\mathcal{L}')[W_{\mathfrak{q}/\mathfrak{q}'}^{-1}]$$

as a summand in the preimage under the map

$$c_{\mathfrak{q}'} : K(\mathcal{G}'^{(\mathfrak{q},b)}_{\lambda_0}) \to C_{\mathfrak{q}'}(\mathcal{L}')[W_{\mathfrak{q}/\mathfrak{q}'}^{-1}].$$

Proof. Goes along the same lines as the proof of Theorem 4.1, replacing Harish-Chandra's bound with the bound (8) given by b, and Kostant's character formula with Proposition 0.6, we deduce that the same formula holds with $2\dim n$ replaced by b. ∎

In conjunction with Proposition 1.5, Theorem 3.4, and corresponding character formulas on $(\mathfrak{g}, K)$ (cf. [4, Theorem 7] and [7] for example) this reduces certain branching problems to the question if the restriction in question lies in the category $\mathcal{C}_{\mathrm{fd}(b)}(\mathfrak{g}', K')$ and supplemental information about the multiplicities violating (9).

Kobayashi [13] gives an overview of branching with respect to restriction to a reductive subpair. In particular he formulates the conjecture (Conjecture C of loc. cit.) that if (G, G') is a reductive symmetric pair and whenever an irreducible unitary representation X of G decomposes discretely infinitesimally when restricted to G', then the multiplicities are finite. Theorem 5.1 is our main motivation in formulating Conjecture 0.1.

For Zuckerman-Vogan's cohomologically induced 'standard modules' $A_{\mathfrak{q}}(\lambda)$ Kobayashi gives a necessary and sufficient criterion when the restriction with respect to a reductive symmetric pair (G, G') decomposes discretely with finite multiplicities [10, 12, 11]. This settles the categorical question alluded to above

in this case. Kobayashi shows in loc. cit. that if the restriction of $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable, then it is even of finite length. Furthermore Zuckerman and Vogan gave a character formula for $A_{\mathfrak{q}}(\lambda)$ [22, Proposition 6.4]. However in this setting Harish-Chandra's global characters would answer the same question.

In principle the finite multiplicity condition might be relaxed and only required for a carefully chosen subset of all composition factors. Then these finite multiplicities may still be determined by the characters. However in view of Kobayashi's Conjecture C of loc. cit., such a treatment should not be necessary, at least in the discretely decomposable case.

6 Localizations of Grothendieck groups

We use the notation of section 4 and let

$$\Lambda \subseteq X(T)$$

denote the sublattice generated by the elements of $\Delta(\mathfrak{u},\mathfrak{t})$. We consider the $\mathbf{C}$-vector space of formal unbounded Laurent series $\mathbf{C}[[\Lambda]]$, i.e. we allow arbitrary (not necessarily finite) linear combinations of elements in Λ. Then $\mathbf{C}[[\Lambda]]$ is no more a ring, but it is a $\mathbf{C}[\Lambda]$-module.

We let $\alpha_1,\dots,\alpha_d$ be elements in Λ, giving rise to a basis of $\mathbf{C}[\Lambda]$, containing the simple roots of T in k with respect to $\mathfrak{n}$, and the property that any $\alpha \in \Delta(\mathfrak{u},\mathfrak{t})$ is a sum of some of the elements $\alpha_1,\dots,\alpha_d$, i.e. $\alpha_1,\dots,\alpha_d$ are the simple roots.

We write W_Λ for the Weyl group of (the possibly non-reduced) root system $\Delta(\mathfrak{u},\mathfrak{t})\cup\Delta(\mathfrak{u}^-,\mathfrak{t})$, which is generated by the reflections w_α for $\alpha\in\Delta(\mathfrak{u},\mathfrak{t})$. Then W_Λ contains $W(K,T)$ and naturally acts on $\mathbf{C}[\Lambda]$ and $\mathbf{C}[[\Lambda]]$.

We assume that we are given a cover

$$\Lambda^{\frac{1}{2}} \to \Lambda$$

consisting of a free $\mathbf{Z}$-module $\Lambda^{\frac{1}{2}}$ of rank $2d$, generated by the square roots of elements in Λ. For any $\alpha\in\Delta(\mathfrak{u},\mathfrak{t})$ we pick one square root

$$\alpha^{\frac{1}{2}} \in \Lambda^{\frac{1}{2}}$$

We write $\mathbf{C}[[\Lambda^{\frac{1}{2}}]]$ for the analogously defined the $\mathbf{C}[\Lambda^{\frac{1}{2}}]$-module of Laurent series.

It comes with two actions of two groups. First of all the action of the Weyl group W_Λ naturally extends to the above modules and rings. Furthermore the Galois group

$$G_\Lambda \cong \{\pm 1\}^r$$

of the cover acts on $\Lambda^{\frac{1}{2}}$ and hence on the above modules and rings. We think of an element $\sigma\in G_\Lambda$ as a collection of signs, i.e.

$$\sigma(\alpha) = \sigma_\alpha\cdot\alpha,$$

where

$$\sigma_\alpha \in \{\pm 1\}.$$

Then G_Λ is generated by the *simple signs* $\sigma_1, \ldots, \sigma_d$ with the property that

$$\sigma_i(\alpha_j) = (1 - 2\delta_{ij}) \cdot \alpha_j$$

for the Kronecker symbol δ_{ij}. The actions of the groups W_Λ and G_Λ commute.

We fix a W_Λ-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\Lambda^{\frac{1}{2}} \otimes_{\mathbf{Z}} \mathbf{R}$, and tend to write the arguments additively, even though we consider the group $\Lambda^{\frac{1}{2}}$ multiplicatively when it comes to Laurent series.

Consider the elements

$$d_{\alpha,\pm} := \alpha^{-\frac{1}{2}} \pm \alpha^{\frac{1}{2}} \in \mathbf{C}[\Lambda^{\frac{1}{2}}],$$

$$s_\alpha := \alpha^{\frac{1}{2}} \sum_{k=0}^{\infty} \alpha^k \in \mathbf{C}[[\Lambda^{\frac{1}{2}}]],$$

and for $n \geq 0$ we set

$$y_\alpha^{(n)} := s_\alpha^n + (-1)^{n+1} w_\alpha s_\alpha^n.$$

Then

$$d_{\alpha,-} \cdot s_\alpha = 1$$

and

$$d_{\alpha,-} \cdot w_\alpha s_\alpha = -1.$$

Therefore, for $n > 0$

$$d_{\alpha,-} \cdot y_\alpha^{(n)} = y_\alpha^{(n-1)}, \tag{10}$$

and we conclude that

$$d_{\alpha,-}^n \cdot y_\alpha^{(n)} = y_\alpha^{(0)} = 0. \tag{11}$$

For any element $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$ we define a map

$$p_\alpha : \mathbf{C}[[\Lambda]] \to \mathbf{C}[[\Lambda]]$$

as follows. Given an element

$$m \in \mathbf{C}[[\Lambda^{\frac{1}{2}}]],$$

we have a unique decomposition

$$m = m_\alpha^+ + m_\alpha^-$$

where

$$m_\alpha^\pm = \sum_{\mu \in \Lambda^{\frac{1}{2}}} c_\mu^\pm \cdot \mu$$

with $c_\mu \in \mathbf{C}$ and

$$c_\mu^\pm = 0$$

if $\mu^2 \in \Lambda$ does not lie in the closure of a Weyl chamber, whose closure contains $\pm\alpha$ (i.e. $\pm\langle \mu, \alpha\rangle < 0$). For those μ contained in the intersection of the closures of Weyl chambers of α and $-\alpha$ (i.e. $\langle \mu, \alpha\rangle = 0$), we insist on

$$c_\mu^- = 0,$$

just to make an explicit choice.

Then the elements $m_\alpha^\pm$ are uniquely determined by m and α and furthermore

$$p_\alpha(m) := s_\alpha \cdot m_\alpha^+ - (w_\alpha s_\alpha) \cdot m_\alpha^-$$

is well defined as well.

Proposition 6.1. *The map p_α is a section of the multiplication by $d_{\alpha,-}$ map*

$$t_\alpha : \mathbf{C}[[\Lambda^{\frac{1}{2}}]] \to \mathbf{C}[[\Lambda^{\frac{1}{2}}]], \quad f \mapsto d_{\alpha,-} \cdot f.$$

In particular the latter is surjective.

Proof. We have for any $m \in \mathbf{C}[[\Lambda^{\frac{1}{2}}]]$

$$d_{\alpha,-} \cdot p_\alpha(m) = (d_{\alpha,-} s_\alpha) \cdot m_\alpha^+ - (d_{\alpha,-} w_\alpha s_\alpha) \cdot m_\alpha^- = 1 \cdot m_\alpha^+ - (-1) \cdot m_\alpha^- = m,$$

showing the claim. □

For any elements $\beta_1, \ldots, \beta_s \in \Lambda$ we denote by $\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\beta_1,\ldots,\beta_s)}$ the subspace of $\mathbf{C}[[\Lambda^{\frac{1}{2}}]]$ which consists of $(\beta_1, \ldots, \beta_s)$-*finite* Laurent series in the following sense:

$$f = \sum_{\lambda \in \Lambda} f_\lambda \lambda$$

is $(\beta_1, \ldots, \beta_s)$-*finite*, if for any $\lambda \in \Lambda^{\frac{1}{2}}$ the set

$$\{(k_1, \ldots, k_s) \in \mathbf{Z}^s \mid f_{\lambda \beta_1^{k_1} \cdots \beta_s^{k_s}} \neq 0\}$$

is finite. We have

$$\mathbf{C}[\Lambda^{\frac{1}{2}}] = \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha_1,\ldots,\alpha_r)}.$$

Lemma 6.2. *For any $n \geq 0$ we have*

$$p_\alpha^n \left(\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} \cdot y_\alpha^{(1)} \right) \subseteq \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} \cdot y_\alpha^{(n+1)}.$$

Proof. We easily reduce to the case of a monomial. We have for any $\mu \in \Lambda^{\frac{1}{2}}$

$$\mu \cdot y_\alpha^{(1)} = \alpha^{\frac{1}{2}} \sum_{k \in \mathbf{Z}} \alpha^k \mu,$$

and as multiplication by μ translates the Weyl chambers, there is a $k_\mu \in \mathbf{Z}$ with the property that

$$\left(\mu \cdot y_\alpha^{(1)} \right)^{\pm} = \mu \alpha^{\pm\frac{1}{2}} \sum_{\pm k \geq \pm k_\mu} \alpha^k,$$

or with the Kronecker symbol $\delta_{1,\pm 1}$

$$\left(\mu \cdot y_\alpha^{(1)}\right)^{\pm} = \delta_{1,\pm 1} \cdot \mu\alpha^{\frac{1}{2}+k_\mu} + \mu\alpha^{\pm\frac{1}{2}} \sum_{\pm k > \pm k_\mu} \alpha^k.$$

We assume that we are in the first case, the argument for the second being same. With this notation

$$p_\alpha^n \left(\mu \cdot y_\alpha^{(1)}\right) = s_\alpha^n \cdot \left(\mu \cdot y_\alpha^{(1)}\right)^+ + (-1)^n (w_\alpha s_\alpha)^n \cdot \left(\mu \cdot y_\alpha^{(1)}\right)^- =$$

$$\mu \cdot \alpha^{k_\mu} \cdot \left(s_\alpha^{n+1} + (-1)^n (w_\alpha s_\alpha)^{n+1}\right) = \mu \cdot \alpha^{k_\mu} \cdot y_\alpha^{(n+1)}.$$

This proves the claim. □

Proposition 6.3. *For any $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$, and for any $n \geq 0$ the kernel of t_α^n is given by*

$$\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} \cdot y_\alpha^{(n)}.$$

Proof. By relation (11) the kernel contains the subspace

$$\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} \cdot y_\alpha^{(n)}.$$

We show by induction on n, that this space indeed contains the kernel. The case $n = 0$ is clear. For the case $n = 1$ we need to show that any element

$$f = \sum_{\lambda \in \Lambda} f_\lambda \lambda \in \ker t_\alpha$$

is of the shape

$$f = g \cdot y_\alpha^{(n)}$$

with

$$g = \sum_{\lambda \in \Lambda} g_\lambda \lambda \in \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)}.$$

We pick a set of representatives $L_\alpha \subseteq \Lambda^{\frac{1}{2}}$ for $\Lambda^{\frac{1}{2}}/\alpha^{\mathbf{Z}}$, and set for any $\lambda_0 \in L_\alpha$ and any $k \in \mathbf{Z}$

$$g_{\lambda_0\alpha^k} := \begin{cases} f_{\lambda_0\alpha^{-\frac{1}{2}}} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

Then

$$g \cdot y_\alpha^{(n)} = \alpha^{\frac{1}{2}} \cdot \sum_{\lambda_0 \in L_\alpha, k, l \in \mathbf{Z}} g_{\lambda_0\alpha^k} \cdot \lambda_0\alpha^k \cdot \alpha^l =$$

$$\alpha^{\frac{1}{2}} \cdot \sum_{\lambda_0 \in L_\alpha, l \in \mathbf{Z}} f_{\lambda_0\alpha^{-\frac{1}{2}}} \cdot \lambda_0\alpha^l = \sum_{\lambda_0 \in L, l \in \mathbf{Z}} f_{\lambda_0\alpha^l} \cdot \lambda_0\alpha^l = f,$$

because saying that f is annihilated by $d_{\alpha,-}$ is the same to say that

$$f_{\lambda\alpha^{-\frac{1}{2}}} = f_{\lambda\alpha^{\frac{1}{2}}}.$$

By construction g is (α)-finite and the claim for $n = 1$ follows. Now assume that the claim is true for a given $n \geq 1$. We have

$$\ker t_\alpha^{n+1} = \ker t_\alpha^n + p_\alpha^n \ker t_\alpha.$$

By Lemma 6.2 we get

$$p_\alpha^n \ker t_\alpha = p_\alpha^n \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} \cdot y_\alpha^{(1)} \subseteq \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} \cdot y_\alpha^{(n+1)}.$$

As

$$d_{\alpha,-} \in \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)}$$

we see with (10) that

$$\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} \cdot y_\alpha^{(n)} \subseteq \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} \cdot y_\alpha^{(n+1)},$$

and we already know that the left hand side is the kernel of t_α^n, concluding the proof. □

For $m \geq 0$ we set

$$d_{\alpha,\pm}^{(m)} := \alpha^{-\frac{m}{2}} \pm \alpha^{\frac{m}{2}} \ \in \ \mathbf{C}[[\Lambda]]_{(\alpha)}.$$

Then the collection of elements

$$d_{\alpha,+}^{(0)}, d_{\alpha,\pm}^{(1)}, d_{\alpha,\pm}^{(2)}, \cdots$$

forms a $\mathbf{C}$-basis of $\mathbf{C}[\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}]$. We have the relations

$$d_{\alpha,+}^{(m)} = \frac{1}{2} d_{\alpha,-}^{(1)} \cdot d_{\alpha,-}^{(m-1)} + \frac{1}{2} d_{\alpha,+}^{(1)} \cdot d_{\alpha,+}^{(m-1)}, \tag{12}$$

and

$$d_{\alpha,-}^{(m)} = d_{\alpha,-}^{(1)} \cdot \sum_{k=0, k\equiv m+1 (\mathrm{mod}\ 2)}^{\frac{m}{2}} d_{\alpha,+}^{(k)}. \tag{13}$$

Relations (12) and (13) reduce the multiplicative structure of $\mathbf{C}[\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}]$ to the two elements $d_{\alpha,\pm}^{(1)}$. Consider the space

$$Y_\alpha^{(n)} := \sum_{k=0}^{n} \mathbf{C} \cdot y_\alpha^{(k)} + \mathbf{C} \cdot d_{\alpha,+} \cdot y_\alpha^{(k)}.$$

Relation (10) shows that

$$d_{\alpha,-} \cdot Y_\alpha^{(n)} = Y_\alpha^{(n-1)}.$$

Analogously we are going to prove

Proposition 6.4. *For any $n > 0$ we have*

$$(\alpha^{-1} + \alpha) \cdot y_\alpha^{(n)} = 2 y_\alpha^{(n)} + y_\alpha^{(n-2)}. \tag{14}$$

Proof. Observe that we have the relation

$$\alpha^{-1} \cdot s_\alpha = \alpha^{-\frac{1}{2}} \sum_{k=0}^{\infty} \alpha^k = \alpha^{-\frac{1}{2}} + s_\alpha.$$

Therefore

$$(\alpha^{-1} + \alpha^1) \cdot s_\alpha = \alpha^{-\frac{1}{2}} - \alpha^{\frac{1}{2}} + 2s_\alpha = d_{\alpha,-} + 2s_\alpha,$$

and

$$(\alpha^{-1} + \alpha^1) \cdot w_\alpha s_\alpha = w_\alpha \left((\alpha^{-1} + \alpha^1) \cdot s_\alpha\right) = -d_{\alpha,-} + 2w_\alpha s_\alpha.$$

Hence

$$(\alpha^{-1} + \alpha) \cdot y_\alpha^{(n)} = (\alpha^{-1} + \alpha^1) \cdot (s_{\alpha,-}^n + (-1)^{n+1} w_\alpha s_\alpha^n) =$$

$$(d_{\alpha,-} + 2s_\alpha) s_\alpha^{n-1} + ((-1)^n d_{\alpha,-} + (-1)^{n+1} 2 w_\alpha s_\alpha) w_\alpha s_\alpha^{n-1} =$$

$$d_{\alpha,-} y_\alpha^{(n-1)} + 2 y_\alpha^{(n)} = y_\alpha^{(n-2)} + 2 y_\alpha^{(n)},$$

thanks to relation (10). □

We denote by $\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\alpha^{\frac{1}{2}}=1}$ a system of representatives for

$$\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} / \langle \alpha^{\frac{1}{2}} - 1 \rangle.$$

Then

Corollary 6.5. *For any $n \geq 0$ the kernel of t_α^n is*

$$\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{(\alpha)} \cdot y_\alpha^{(n)} = \sum_{k=0}^{n} \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\alpha^{\frac{1}{2}}=1} \cdot y_\alpha^{(k)} \; + \; \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\alpha^{\frac{1}{2}}=1} \cdot d_{\alpha,+} \cdot y_\alpha^{(k)}.$$

Proof. As

$$d_{\alpha,+}^2 = 2 + (\alpha^{-1} + \alpha),$$

Proposition 6.4 shows in particular that $Y_\alpha^{(n)}$ is a $\mathbf{C}[\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}]$-module and the factor module

$$Y_\alpha^{(n)} / Y_\alpha^{(n-1)}.$$

is a vector space of dimension 2, spanned by the classes of

$$y_\alpha^{(n)}, d_{\alpha,+} y_\alpha^{(n)}.$$

Now the relation

$$2\alpha^{\frac{1}{2}} = d_{\alpha,+} - d_{\alpha,-}$$

shows that

$$2\alpha^{\frac{1}{2}} \cdot y_\alpha^{(n)} = d_{\alpha,+} \cdot y_\alpha^{(n)} - y_\alpha^{(n-1)},$$

and a similar relation holds for $2\alpha^{-\frac{1}{2}}$, which shows that we may find a representation as claimed in the corollary. □

In the sequel we always assume that each element

$$x = \sum_\lambda x_\lambda \lambda \ \in \ \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\alpha^{\frac{1}{2}}=1}$$

in our system of representatives satisfies the boundedness condition

$$0 \leq \frac{\langle \lambda - \lambda_0, \alpha \rangle}{\langle \alpha, \alpha \rangle} < \frac{1}{2} \tag{15}$$

whenever $x_\lambda \neq 0$ with respect to a fixed $\lambda_0 \in \Lambda^{\frac{1}{2}}$.

In our next step we generalize the results so far obtained for the univariate case to the multivariate situation. Let $\alpha_1, \dots, \alpha_r \in \Delta(\mathfrak{u}, \mathfrak{t})$ be pairwise distinct, and choose positive integers $n_1, \dots, n_r$, and write

$$t_{\underline{\alpha}}^{\underline{n}} := t_{\alpha_1}^{n_1} \circ t_{\alpha_2}^{n_2} \circ \cdots \circ t_{\alpha_r}^{n_r},$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ and similarly $\underline{n}$ is the corresponding tuple of integers. The multiplication maps t_{α_i} commute, and we let for any $1 \leq i \leq r$ denote by $\underline{\alpha}^{\{i\}}$ and $\underline{n}^{\{i\}}$ the tuples where the i-th component has been removed. More generally for any subset $I \subseteq \{1, \dots, r\}$ we denote by $\underline{\alpha}^I$ and $\underline{n}^I$ the resulting tuples where the entries with indices occuring in I are deleted. Introduce the elements

$$y_{\underline{\alpha}}^{\underline{n}} := s_{\alpha_1}^{n_1} \cdot s_{\alpha_2}^{n_2} \cdots s_{\alpha_r}^{n_r} + (-1)^{1+n_1+\cdots+n_r} (w_{\alpha_1} s_{\alpha_1})^{n_1} \cdot (w_{\alpha_2} s_{\alpha_2})^{n_2} \cdots (w_{\alpha_r} s_{\alpha_r})^{n_r}$$

Proposition 6.6. *For any $\underline{\alpha}$ and any $\underline{n}$ we have*

$$\ker t_{\underline{\alpha}}^{\underline{n}} = \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\underline{\alpha}} \cdot y_{\underline{\alpha}}^{\underline{n}} \ + \ \sum_{i=1}^r \ker t_{\underline{\alpha}^{\{i\}}}^{\underline{n}^{\{i\}}}.$$

We let $\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\underline{\alpha}^I=1}$ denote a system of representatives for

$$\mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\underline{\alpha}^I} / \langle \alpha_i^{\frac{1}{2}} - 1 | i \notin I \rangle,$$

not necessarily subject to condition (15). As before we have

Corollary 6.7. *The kernel of $t_{\underline{\alpha}}^{\underline{n}}$ is given by*

$$\sum_{I \subsetneq \{1,\dots,r\}} \sum_{J \subseteq \overline{I}} \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\underline{\alpha}^I=1} \cdot y_{\underline{\alpha}^I}^{\underline{n}^I} \cdot \prod_{j \in J} d_{\alpha_j,+},$$

where $\overline{I}$ denotes the complement of I.

We remark that by taking G_Λ-invariants, we immediately get a description of the kernel restricted to $\mathbf{C}[[\Lambda]]$. Furthermore it is not hard to take the action of the Weyl group into account as well. However we will not need to make this more precise.

Proof. We prove Proposition 6.6 by induction on r, and also make use of the corollary in the cases guaranteed by the induction hypothesis. Consequently we assume the claim to be true for all $r' < r$ and let $z_0 \in \ker t_{\underline{\alpha}}^{\underline{n}}$. Then

$$d_{\alpha_1,-}^{n_1} \cdot z_0 \ \in \ \ker t_{\underline{\alpha}^{\{1\}}}^{\underline{n}^{\{1\}}}.$$

For the sake of readability we set for any $J \subseteq I \subseteq \{1, \dots, r\}$, $\emptyset \subsetneq I$,

$$y_{\underline{\alpha},I}^{\underline{n},J} := y_{\underline{\alpha}^{\overline{I}}}^{\underline{n}^{\overline{I}}} \cdot \prod_{j \in J} d_{\alpha_j,+}.$$

By the induction hypothesis we find for any $J \subseteq I \subsetneq \{2, \dots, r\}$, $I \neq \emptyset$, elements

$$c_{I,1}^{J} \in \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\underline{\alpha}^{\overline{I}}=1}$$

with the property that

$$d_{\alpha_1,-}^{n_1} \cdot z_0 \ = \sum_{\emptyset \subsetneq I \subseteq \{2,\dots,r\}} \sum_{J \subseteq I} c_{I,1}^{J} \cdot y_{\underline{\alpha},I}^{\underline{n},J}.$$

By the established relations we may assume that our chosen system of representatives satisfies condition (15) for any $i \notin I$ for $\lambda_0 = 0$. We set

$$\tilde{z}_1 := \sum_{\emptyset \subsetneq I \subsetneq \{2,\dots,r\}} \sum_{J \subseteq I} p_{\alpha_1}^{n_1}(c_{I,1}^{J} \cdot y_{\underline{\alpha},I}^{\underline{n},J}),$$

and

$$z_1 := \sum_{J \subseteq \{2,\dots,r\}} p_{\alpha_1}^{n_1}(c_{\{2,\dots,r\},1}^{J} \cdot y_{\underline{\alpha},\{2,\dots,r\}}^{\underline{n},J}).$$

Then $z_0 - z_1 - \tilde{z}_1$ is in the kernel of $t_{\alpha_1}^{n_1}$, therefore we find $a_1, b_1 \in \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\alpha_1=1}$ with

$$z_0 = z_1 \ + \ \tilde{z}_1 \ + \ b_1 \cdot y_{\alpha_1}^{(n_1)} \ + \ a_1 \cdot y_{\alpha_1}^{(n_1)} \cdot d_{\alpha_1,+}.$$

Furthermore we note that the induction hypothesis applies to each summand of $\tilde{z}_1$, which means that we may assume that the elements $c_{I,1}^{J}$ are α_1-finite for $I \subsetneq \{2, \dots, r\}$. We need to show that the same holds for the remaining cases, i.e. the summands of z_1. This will allow us to replace the section $p_{\alpha}^{n_1}$ by a mere multiplication, as then the sum of the elements

$$c_{I,1}^{J} \cdot y_{\underline{\alpha},I \cup \{1\}}^{\underline{n},J}$$

is a preimage of $d_{\alpha_1,-}^{n_1} \cdot z_0$ under $t_{\alpha_1}^{n_1}$, and adapting a_1, b_1, and the remaining $c_{I,1}^{J}$ again, we eventually find a representation as claimed.

If $r = 1$ we are done, as then the elements under consideration are 0, and in particular α_1-finite. So we may assume that $r > 1$.

If for some $1 < i \leq r$ we have $\langle \alpha_1, \alpha_i \rangle \neq 0$ we are done as well, as condition (15) for α_i implies that the coefficients in question are α_1-finite as well.

This reduces us to the case that the roots $\alpha_1, \dots, \alpha_r$ are all pairwise orthogonal, because if there exists a pair of non-orthogonal roots, we may label one of them as α_1, and proceed as before.

This pairwise orthogonality implies that the sections p_{α_i} commute with multiplications by elements whose monomials are supported only on α_j for $j \neq i$.

We define

$$d_{\alpha_2,-}^{n_2} \cdot z_1 \;=\; \sum_{\emptyset \subsetneq I \subseteq \{1,3,\dots,r\}} \sum_{J \subseteq I} c_{I,2}^J \cdot y_{\underline{\alpha},I}^{\underline{n},J},$$

with corresponding representatives $c_{I,2}^J$ as guaranteed by the induction hypothesis, and

$$\tilde{z}_2 := \sum_{\emptyset \subsetneq I \subsetneq \{1,3,\dots,r\}} \sum_{J \subseteq I} p_{\alpha_2}^{n_2}(c_{I,2}^J \cdot y_{\underline{\alpha},I}^{\underline{n},J}).$$

and

$$z_2 := \sum_{J \subseteq \{1,3,\dots,r\}} p_{\alpha_2}^{n_2}(c_{\{1,3,\dots,r\},2}^J \cdot y_{\underline{\alpha},\{1,3,\dots,r\}}^{\underline{n},J}).$$

Then again we have representatives a_2, b_2 with

$$z_1 = z_2 \;+\; \tilde{z}_2 \;+\; b_2 \cdot y_{\alpha_2}^{(n_2)} \;+\; a_2 \cdot y_{\alpha_2}^{(n_2)} \cdot d_{\alpha_2,+},$$

where again the induction hypothesis applies to all summands of $\tilde{z}_2$ which allows us to assume the α_2-finiteness of the corresponding coefficients. Consequently we may assume that for $2 \in I \subsetneq \{1, \dots, r\}$ and any $J \subseteq I$ there is an element

$$\tilde{c}_{I,2}^J \in \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\underline{\alpha}^I = 1}$$

with the property that

$$\tilde{z}_2 + b_2 \cdot y_{\alpha_2}^{(n_2)} + a_2 \cdot y_{\alpha_2}^{(n_2)} \cdot d_{\alpha_2,+} = \sum_{2 \in I \subsetneq \{1,\dots,r\}} \sum_{J \subseteq I} \tilde{c}_{I,2}^J \cdot y_{\underline{\alpha},I}^{\underline{n},J}.$$

This, together with the orthogonality relation, yields the identity

$$\begin{aligned} z_2 = & \sum_{J \subseteq \{2,\dots,r\}} p_{\alpha_1}^{n_1}(c_{\{2,\dots,r\},1}^J) \cdot y_{\underline{\alpha},\{2,\dots,r\}}^{\underline{n},J} \\ & - \sum_{2 \in I \subsetneq \{1,\dots,r\}} \sum_{J \subseteq I} \tilde{c}_{I,2}^J \cdot y_{\underline{\alpha},I}^{\underline{n},J}, \end{aligned}$$

where all coefficients on the right hand side are α_2-finite, and even satisfy the stronger condition (15). This implies that the coefficients occuring in z_i are α_i-finite as well. The argument goes as follows. Multiplying the above identity with

$$d := \prod_{j \neq 2} d_{j,-}^{n_j},$$

and substituting the definition of z_2, we get

$$\sum_{J \subseteq \{1,3\dots,r\}} p_{\alpha_2}^{n_2}(c_{\{1,3,\dots,r\},2}^J) \cdot \prod_{2 \neq j} d_{\alpha_j,-}^{n_j} \cdot \prod_{j \in J} d_{\alpha_j,+} =$$

$$\sum_{J\subseteq\{2,\ldots,r\}} c^J_{\{2,\ldots,r\},1}\cdot\prod_{1\neq j\neq 2} d^{n_j}_{\alpha_j,-}\cdot\prod_{j\in J} d_{\alpha_j,+}\cdot y^{(n_2)}_{\alpha_2}$$
$$-\sum_{2\in I\subsetneq\{1,\ldots,r\}}\sum_{J\subseteq I}\tilde{c}^J_{I,2}\cdot\prod_{2\neq j\notin I} d^{n_j}_{\alpha_j,-}\cdot\prod_{j\in J} d_{\alpha_j,+}\cdot y^{(n_2)}_{\alpha_2}.$$

By the very definition of the section p_{α_2}, and the fact that by (15) the coefficients $c^J_{\{1,3,\ldots,r\},2}$ are uniquely determined by the left hand side of this equation, this shows the α_2-finiteness of every $c^J_{\{1,3,\ldots,r\},2}$, concluding the proof. $\square$

Proposition 6.8. *For any $n\geq 1$, $\alpha\in\Delta(\mathfrak{u},\mathfrak{t})$, we have*

$$s^n_\alpha=\sum_{k=0}^{\infty}\binom{n-1+k}{n-1}\cdot\alpha^{\frac{n}{2}+k},$$

and

$$d_{\alpha,+}\cdot s^n_\alpha=\sum_{k=0}^{\infty}\frac{n-1+2k}{n-1+k}\cdot\binom{n-1+k}{n-1}\cdot\alpha^{\frac{n-1}{2}+k},$$

subject to the convention that for $n=1$ and $k=0$

$$\frac{n-1+2k}{n-1+k}=1.$$

Proof. We omit the proof, a straightforward induction on n. $\square$

Theorem 6.9. *Let*

$$z=\sum_\lambda z_\lambda\cdot\lambda\ \in\ \ker t^{\underline{n}}_{\underline{\alpha}}$$

with the property that there exists a $\lambda_0\in\Lambda^{\frac{1}{2}}$ such that for any $1\leq i\leq r$ and any $\lambda\in\Lambda^{\frac{1}{2}}$ with

$$|\langle\lambda-\lambda_0,\alpha_i\rangle|<\frac{n_i+1}{2}\cdot\langle\alpha_i,\alpha_i\rangle+\sum_{j\neq i}\frac{n_j}{2}\cdot\langle\alpha_i,\alpha_j\rangle,\tag{16}$$

we have

$$z_\lambda=0$$

then

$$z=0.$$

Proof. We may assume that z is *primitive*, i.e. does not lie in any of the kernels $\ker t^{\underline{n}^{\{i\}}}_{\underline{\alpha}^{\{i\}}}$ for $1\leq i\leq r$. Now for any i we consider

$$z^{(i)}:=t^{\underline{n}^{\{i\}}}_{\underline{\alpha}^{\{i\}}}(z),$$

which is primitive in $\ker t^{n_i}_{\alpha_i}$, and hence of the form

$$z^{(i)}\ =\ \sum_{k=1}^{n_i} a_k\cdot y^{(k)}_\alpha\ +\ b_k\cdot d_{\alpha,+}\cdot y^{(k)}_\alpha$$

by Corollary 6.5, with

$$a_1, b_1, \ldots, a_{n_i}, b_{n_i} \in \mathbf{C}[[\Lambda^{\frac{1}{2}}]]_{\alpha^{\frac{1}{2}}=1},$$

where we may assume that our system of representatives satisfies condition (15) for λ_0 as in the statement of the theorem. By Proposition 6.8 and our choice of representatives we know that if we write

$$z^{(i)} = \sum_{\mu} z_{\mu}^{(i)} \cdot \lambda$$

with $z_{\mu}^{(i)} \in \mathbf{C}$, then $z_{\mu}^{(i)} \neq 0$ for some μ satisfying

$$|\langle \mu - \lambda_0, \alpha_i \rangle| \; < \; \frac{n_i + 1}{2} \cdot \langle \alpha_i, \alpha_i \rangle.$$

This in turn implies the existence of a λ with $z_\lambda \neq 0$ subject to the condition

$$|\langle \lambda - \lambda_0, \alpha_i \rangle| \; < \; \frac{n_i + 1}{2} \cdot \langle \alpha_i, \alpha_i \rangle + \sum_{j \neq i} \frac{n_j}{2} \cdot \langle \alpha_i, \alpha_j \rangle.$$

□

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